1.1 Let \( A = \{a_1, \ldots, a_m\} \); since \( f \) is 1-to-1, \& all the \( a_i \)'s are distinct, the \( f(a_i) \)'s are distinct (more explicitly, \( f(a_i) = f(a_j) \iff a_i = a_j \) (by 1-to-1)). Hence \( f(A) := \{f(a_1), f(a_2), \ldots, f(a_m)\} \subseteq B \) has \( m \) distinct elements, \& since a set always has at least as many elements as any one of its subsets, we have

\[ n = \#(B) \geq \#(f(A)) = m \]

1.2 If \( f: A \to B \) is bijective, then it is injective, hence by

\[ (1.1) \quad m \leq n \]

Moreover, if \( f: A \to B \) is bijective, then \( f \) admits an inverse map \( f^{-1}: B \to A \) which is also bijective. Since \( f^{-1} \) is bijective, it is injective, hence, applying (1.1) again (taking \( f^{-1} \) in place of \( f \), \& switching the roles of \( A \) \& \( B \) \& \( n \leq m \)), we get an inequality:

\[ n = \#(B) \leq \#(A) = m \quad \text{(i.e.,} \quad n \leq m) \]

Combining the boxed inequalities, we get \( n = m \).

2.1 A function \( f: A \to B \) is uniquely determined by listing the values \( f(a) \in B \), for each \( a \in A \). In other words, one can find all the functions \( \{a_1, a_2, a_3\} \to \{1, 2, 3\} \) by determining all ways to assign a number 1, 2, or 3 to \( a_1 \), then assign a number 1, 2, or 3 to \( a_2 \), \& then finally assign a number 1, 2, or 3 to \( a_3 \).

There are three ways to make such an assignment for \( a_1 \),

\[ \overline{a_1} \]

\[ \overline{a_2} \]

\[ \overline{a_3} \]

Since

The assignment for \( a_1 \) doesn't affect the assignment for \( a_2 \), there is a total

\[ 3 \cdot 3 = 9 \]

ways to make an assignment for \( a_2 \) \& then make an assignment for \( a_2 \). The assignment for \( a_1 \) \& \( a_2 \) don't affect the assignment for \( a_3 \), there is
2.1 (cont.)

a total of $9 \cdot 3 = 27$ ways to assign $a_1, a_2,$ and then assign $a_3$. Thus, there are $27$ functions $\{a_1, a_2, a_3\} \rightarrow \{1, 2, 3\}$.

2.2 We follow the same process as 2.1, but in this case, we must make a total of four assignments, and each assignment allows only two choices. Thus we get:

\[
\begin{array}{c}
\text{two choices for } a_1 \\
\text{two choices for } a_2 \\
\text{two choices for } a_3 \\
\end{array}
\]

\[
2 \cdot 2 \cdot 2 = 8
\]

functions $\{a_1, a_2, a_3, a_4\} \rightarrow \{1, 2, 3, 4\}$.

2.3 Careful counting allows us to generalize the specific cases above to arbitrary finite sets $A$ & $B$. I.e., if $n=\#(A)$ & $m=\#(B)$, then to define a function, for each $a \in A$ we must assign an element of $B$. For each fixed $a \in A$ there are $n=\#(B)$ many choices for this assignment, thus to define a function we must make $m=\#(A)$ independent assignments (i.e., the choice made for a given element of $A$ does not affect the assignments of other elements), & each assignment can be done in $n=\#(B)$ many ways. Hence, there are

\[
\underbrace{n \cdot n \cdots n} = n^m
\]

functions $A \rightarrow B$. 

3.1
\[ f(1,0)=1; \ f(2,1)=2; \ f(1,2)=3; \ f(1,3)=6; \ f(3,2)=5; \ f(3,1)=4 \]

3.2  
(a) \( (n+m)=(n'+m') \).

If \( f(n,m)=f(n',m') \) — i.e., \( \frac{1}{2} (n+m-2)(n+m-1)+m=\frac{1}{2} (n'+m'-2)(n'+m'-1)+m' \), then

\[ \frac{1}{2} (n+m-2)(n+m-1)-(n'+m'-2)(n'+m'-1) = m'-m. \]

But \( (n+m)=(n'+m') \), so the left-hand side of this equation must be zero. Hence \( 0=m'-m \), i.e.,

\[ m=m'. \]

Now, since \( (n+m)=(n'+m') \) & \( m=m' \), it follows immediately that \( n=n' \) (substitute \( m' \) for \( m \) in \( (n+m)=(n'+m') \) & solve for \( n \)). Thus, under assumption \( (n+m)=(n'+m') \), \( f(n,m)=f(n',m') \Rightarrow (m=m' & n=n') \Rightarrow (m,m)=(n',m') \).

(b) If \( n=n' \) & \( n+m\neq n'+m' \), suppose \( f(n,m)=f(n',m') \), then we have

\[ \frac{1}{2} (n+m-2)(n+m-1)+m=\frac{1}{2} (n'+m'-2)(n'+m'-1)+m' \]

\[ \frac{1}{2} (n+m-2)(n+m-1)-(n'+m'-2)(n'+m'-1) \]

\[ = m=m'. \]  

Now, since \( n+m\neq n'+m' \), either \( n+m \neq n'+m' \) or else \( n+m = n'+m' \).

If \( n+m = n'+m' \), then (since \( n=n' \)) \( m=m' \), so the left-hand side of the underlined equation is strictly positive. Since \( n+m \neq n'+m' \), however, the left-hand side must be strictly negative. \( 0 \leq n+m-2 \leq n'+m'-2 \) \& \( 0 \leq n'+m'-1 \leq n'+m'-1 \) imply that \( (n+m-2)(n+m-1) \leq (n'+m'-2)(n'+m'-1) \). Thus, we have reached a contradiction (it's impossible for a strictly positive number to be equal to a strictly negative number). It follows then that under assumption b), \( f(n,m)=f(n',m') \) is \underline{false}.

If \( n+m \neq n'+m' \), then we deduce that the RHS is strictly negative, but the LHS is strictly positive, using identical reasoning.
3.2c) Assume \( m = m' \) & \( n m \neq n' + m' \). Suppose \( f(n,m) = f(n',m') \), then

\[
\frac{1}{2} (n+m-2)(n+m-1) + m = \frac{1}{2} (n'+m'-2)(n'+m'-1) + m' \\
(n+m-2)(n+m-1) - (n'+m'-2)(n'+m'-1) = 2(m-m') = 0
\]

since \( m = m' \).

Now since \( n m \neq n' + m' \), we have either \( n m \neq n' + m' \) or else \( n m \neq n' + m' \). In either case, using reasoning similar to b), we get that \( (n+m-2)(n+m-1) - (n'+m'-2)(n'+m'-1) \neq 0 \) (as it will be strictly positive or strictly negative). Thus \( f(n,m) = f(n',m') \) leads to a contradiction (under assumption c)), & hence \( f(n,m) = f(n',m') \) is \underline{False}, so

\[ f(n,m) = f(n',m') \implies (n,m) = (n',m') \] is \underline{True}.

3.3

\[ f(3,3) = 13, \quad f(3,4) = 19, \quad f(1,7) = 28 \]

3.4 \[ f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \] is the inverse of the map that enumerates \( \mathbb{N} \times \mathbb{N} \) as indicated in the picture.