MATH 4281 Exam I   Name: ____________________

Question 1. True of False . . . . 20 points

Mark each of the following "T" (if true) of "F" (if false).

2 pts (a) __**F**__ If $G = \emptyset$ and $*$ is the unique operation on G, then $(G, *)$ is a group.

3 pts (b) __**F**__ $(\mathbb{N}, +)$ is a subgroup of $(\mathbb{Z}, +)$.

3 pts (c) __**T**__ $(\mathbb{N}^\times, \cdot)$ is a subgroup of $(\mathbb{Z}^\times, \cdot)$.

3 pts (d) __**F**__ For $n \in \mathbb{Z}_{>0}$, S_n is *never* an abelian group.

3 pts (e) __**F**__ Every cyclic group is isomorphic to $\mathbb{Z}_n$ for some $n \in \mathbb{Z}_{>0}$.

3 pts (f) __**T**__ If G is an abelian group, then $G = Z(G)$ where

$$Z(G) := \{g \in G \mid gh = hg \quad \forall h \in G\}.$$

3 pts (g) __**F**__ If G is a group and $H, K \subset G$ are subgroups, then $HK \subset G$ is also a subgroup where $HK = \{hk \in G \mid h \in H, k \in K\}$.

Question 2. More True of False (Tricky!) . . . . 20 points

Mark each of the following "T" (if true) of "F" (if false).

4 pts (a) __**F**__ For a group G, there exists a homomorphism $f : \mathbb{Z}_6 \to G$ if and only if there exists an element $g \in G$ of order 6.

4 pts (b) __**F**__ For a group G and two elements $a, b \in G$, there *always* exists a homomorphism $f : \mathbb{Z} \times \mathbb{Z} \to G$ such that $a, b \in f(\mathbb{Z} \times \mathbb{Z})$.

4 pts (c) __**T**__ Let G, H be groups and $f : G \to H$ be an epimorphism. If H is abelian, then $\phi : G \to H : g \mapsto f(g^2)$ is *always* a homomorphism.

4 pts (d) __**T**__ Let G, H be groups and $f : G \to H$ be an epimorphism. If H is *not* abelian, then $\phi : G \to H : g \mapsto f(g^2)$ is *never* a homomorphism.

4 pts (e) __**F**__ Let G, H be groups and $f : G \to H$ be a homomorphism. Then $f(Z(G)) \subset Z(H)$.

Question 3. Find the Orders .... 20 points

Find the order of each element.

2 pts	(a)	∞	$1 \in \mathbb{Z}$
2 pts	(b)	**7**	$3 \in \mathbb{Z}_7$
2 pts	(c)	**2**	$f \in \text{Aut}(\mathbb{C}^2)$ where $f(a,b) = (b,a)$ for any $a, b \in \mathbb{C}$
3 pts	(d)	**12**	$(1723)(465) \in S_9$
3 pts	(e)	**6**	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 8 & 7 & 1 & 5 & 2 & 9 & 3 & 6 \end{pmatrix} \in S_9$
4 pts	(f)	**5**	$\sigma^{42} \in S_{10}$ where $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 5 & 8 & 9 & 10 & 1 & 7 & 3 & 6 \end{pmatrix} \in S_{10}$
4 pts	(g)	**30**	$f \in \text{Aut}(\mathbb{C}^8)$ where

$$f(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = (-a_2, -a_3, -a_4, -a_5, -a_1, a_7, a_8, a_6)$$

Question 4. Symmetric Group .... 20 points

Let us define $\alpha, \beta, \gamma, \delta \in S_7$ to be

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 1 & 7 & 6 & 5 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 4 & 7 & 5 & 6 \end{pmatrix},$$
$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 7 & 3 & 5 & 6 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix}.$$

(a) Describe each of the following elements in either two-line array or cycle notation.

4 pts — i. $\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 4 & 1 & 5 & 7 & 6 \end{pmatrix} = (134)(67)$

4 pts — ii. $\gamma \circ \delta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 1 & 2 & 4 & 7 & 3 & 5 \end{pmatrix} = (1632)(57)$

4 pts — iii. $\beta^7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 4 & 7 & 5 & 6 \end{pmatrix} = (123)(576)$

4 pts — (b) Find $\{\alpha, \beta, \gamma, \delta\} \cap A_7$. $\{\beta, \gamma, \delta\}$

4 pts — (c) Find an element $\epsilon \in S_7$ which is a product of disjoint *transpositions* such that $\epsilon \circ \beta = \beta \circ \epsilon$.

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 6 & 4 & 1 & 3 & 2 \end{pmatrix} = (15)(27)(36)$

Question 5. Semidirect Product .... **20 points**

Let G be a group with multiplicative notation (i.e. the operation on G is given by $(g,h) \mapsto gh$) and H be its subgroup. We define a new operation $*$ on the set $G \times H$ by

$$*\colon (G \times H) \times (G \times H) \to G \times H\colon ((a,b),(c,d)) \mapsto (abcb^{-1}, bd),$$

i.e. $(a,b) * (c,d) = (abcb^{-1}, bd)$. Note that the operation $*$ is *not* the same as the usual operation on the product group $G \times H$, i.e. $((a,b),(c,d)) \mapsto (ac, bd)$.

5 pts (a) Prove that $*$ is associative.

Solution: For any $a, b, c \in G, d, e, f \in H$, we have

$$\begin{aligned}((a,d)*(b,e))*(c,f) &= (adbd^{-1}, de)*(c,f) = (adbd^{-1}dec(de)^{-1}, def)\\ &= (adbece^{-1}d^{-1}, def),\\ (a,d)*((b,e)*(c,f)) &= (a,d)*(bece^{-1}, ef) = (adbece^{-1}d^{-1}, def).\end{aligned}$$

Thus $((a,d)*(b,e))*(c,f) = (a,d)*((b,e)*(c,f))$ and $*$ is associative.

5 pts (b) Prove that $(G \times H, *)$ has an identity and find what it is.

Solution: For any $g \in G$ and $h \in H$, we have

$$(g,h)*(e,e) = (gheh^{-1}, h) = (g,h), \quad (e,e)*(g,h) = (g,h).$$

Thus $(e,e) \in G \times H$ is the identity of $*$.

5 pts (c) Prove that every $(g,h) \in (G \times H, *)$ has an inverse and find what it is.

Solution: For any $g \in G$ and $h \in H$, we have

$$\begin{aligned}(g,h)*(h^{-1}g^{-1}h, h^{-1}) &= (ghh^{-1}g^{-1}hh^{-1}, hh^{-1}) = (e,e),\\ (h^{-1}g^{-1}h, h^{-1})*(g,h) &= (h^{-1}g^{-1}hh^{-1}gh, h^{-1}h) = (e,e).\end{aligned}$$

Thus $(h^{-1}g^{-1}h, h^{-1})$ is the inverse of (g,h) with respect to $*$.

Therefore, $(G \times H, *)$ is a group.

5 pts (d) Prove that the map $f\colon (G \times H, *) \to G\colon (g,h) \mapsto gh$ is a homomorphism.

Solution: For any $g, g' \in G$ and $h, h' \in H$, we have

$$f((g,h)*(g',h')) = f(ghg'h^{-1}, hh') = ghg'h^{-1}hh' = ghg'h' = f((g,h))f((g',h')).$$

Thus f is a homomorphism.

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