MATH 4281 Exam II **Name:** ______________________

Question 1. True of False .... 20 points

Mark each of the following "T" (if true) of "F" (if false).

2 pts (a) **T** Let G be a group and H be a subgroup of G. If $(G : H) = 2$, then H is a normal subgroup of G.

3 pts (b) **F** Let G be a finite group such that every subgroup of G is normal. Then G is abelian.

3 pts (c) **T** If G is a group of order 6, then G should be isomorphic to either $\mathbb{Z}_6$ or S_3.

3 pts (d) **F** If G is a group and $H, K \subset G$ are subgroups. If either H or K is a normal subgroup of G, then HK is also a normal subgroup of G.

3 pts (e) **F** Let G be a group and H be a normal subgroup of G. If both G/H and H are finitely generated abelian groups, then G is also a finitely generated abelian group.

3 pts (f) **T** For $n \in \mathbb{Z}_{\geq 3}$, D_n is *never* an abelian group.

3 pts (g) **F** For $n \in \mathbb{Z}_{\geq 3}$, assume that S_n acts on a set X. If the action is not trivial, i.e. there exists $\sigma \in S_n$ and $x \in X$ such that $\sigma \cdot x \neq x$, then this action is always faithful.

Question 2. Finite abelian groups .... 20 points

Find, <u>*up to isomorphism*</u> and <u>*without repetition*</u>, all abelian groups of given order. Also, find the maximum possible order for some element of each group. You *do not* need to justify your answer.

4 pts (a) 1

Answer. $(\mathbb{Z}_1, 1)$

4 pts (b) 81

Answer. $(\mathbb{Z}_{81}, 81), (\mathbb{Z}_3 \times \mathbb{Z}_{27}, 27), (\mathbb{Z}_9 \times \mathbb{Z}_9, 9),$
$(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9, 9), (\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, 3)$

4 pts (c) 210

Answer. $(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7, 210)$

4 pts (d) 360

Answer. $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, 30), (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5, 90),$
$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, 60), (\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5, 180),$
$(\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, 1230), (\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5, 360)$

4 pts (e) 1000

Answer. $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5, 10),$
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_{25}, 50), (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{125}, 250),$
$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5, 20),$
$(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_{25}, 100), (\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{125}, 500),$
$(\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5, 40),$
$(\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_{25}, 200), (\mathbb{Z}_8 \times \mathbb{Z}_{125}, 1000)$

Question 3. Burnside's lemma .... 20 points

The edges of a regular n-gon are to be painted with r different colors, allowed to use each of which several times. Let X be a set of all such paintings.

(a) Let $n = 6$ and $r = 3$. We want to identify paintings of the edges *up to rotation* with orbits of some group G acting on X.

2 pts i. What is G in this case?

> **Answer.** $\{e, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ where ρ is rotation by 60 degree.

5 pts ii. For each $g \in G$, find the size of X^g, i.e. the number of fixed points in X by g.

> **Answer.** $|X^e| = 729, |X^\rho| = |X^{\rho^5}| = 3, |X^{\rho^2}| = |X^{\rho^4}| = 9, |X^{\rho^3}| = 27$

3 pts iii. Calculate the number of G-orbits in X.

> **Answer.** 130

(b) This time let $n = 4$ and $r = 4$. We want to identify paintings of the edges *up to rotation and reflection* with orbits of some group G acting on X.

2 pts i. What is G in this case?

> **Answer.** Dihedral group $\{e, \rho, \rho^2, \rho^3, r_1, r_2, r_3, r_4\}$ where ρ is rotation by 90 degree and r_1, r_3 (resp. r_2, r_4) are reflections whose axes are perpendicular to some edge (resp. whose axes are diagonals of the given square)

5 pts ii. For each $g \in G$, find the size of X^g, i.e. the number of fixed points in X by g.

> **Answer.** $|X^e| = 256, |X^\rho| = |X^{\rho^3}| = 4,$
> $|X^{\rho^2}| = |X^{r_2} = |X^{r_4}| = 16, |X^{r_1}| = |X^{r_3}| = 64$

3 pts iii. Calculate the number of G-orbits in X.

> **Answer.** 55

You *do not* need to justify your answer.

Question 4. Class equation .... **20 points**

[5 pts] (a) Let G be a finite group acting on a finite set X. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the orbits of G in X such that $|\mathcal{O}_1|, \ldots, |\mathcal{O}_r| > 1$. Prove the following equation

$$|X| = |X^G| + |\mathcal{O}_1| + |\mathcal{O}_2| + \cdots + |\mathcal{O}_r|$$

where $X^G = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$.

[5 pts] (b) Let G be a finite group and $Z(G)$ be its center. Prove the following "class equation"

$$|G| = |Z(G)| + k_1 + k_2 + \cdots + k_r$$

where $k_1, k_2, \ldots, k_r$ are the sizes of conjugacy classes each of which does not consist of a single element.

[5 pts] (c) Let G be a finite group acting on a finite set X. If $\mathcal{O} \subset X$ is a G-orbit, then prove that the order of G is the multiple of the size of $\mathcal{O}$.

[5 pts] (d) Suppose that $|G| = p^k$ for some prime p and $k \in \mathbb{Z}_{>0}$. If G acts on a finite set X, then prove that $|X| \equiv |X^G| \pmod p$ where X^G is defined as in (a).

Answer.

(a) For $x \in X$, $G \cdot x = \{x\}$ if and only if $g \cdot x = x$ for all $g \in G$ if and only if $x \in X^G$. Since X is a disjoint union of all the G-orbits in X, the result follows.

(b) It directly follows from (a) if we let $X = G$ and let G act on G by conjugation.

(c) Since there is an isomorphism $G/\mathrm{Stab}_G(x) \simeq \mathcal{O}$ of G-sets where x is some element in $\mathcal{O}$, we have $|G|/|\mathrm{Stab}_G(x)| = |\mathcal{O}|$. It follows that $|\mathcal{O}|$ is a divisor of $|G|$.

(d) It follows from (a) if we take the equation modulo p. Note that in this case $|\mathcal{O}| > 1$ if and only if p is a divisor of $|\mathcal{O}|$ by (c).

Question 5. Group of order p^3 .... 20 points

Let G be a group such that $|G| = p^3$ for some prime p. Here we prove that $|Z(G)|$ equals p or p^3, where $Z(G)$ is the center of G.

5 pts (a) First suppose that $|Z(G)| = p^3$. In this case find all possible G up to isomorphism.

5 pts (b) From now on we assume that $|Z(G)| \neq p^3$. Prove that $|Z(G)|$ equals p or p^2. (Hint: you may use "class equation".)

5 pts (c) Suppose that $|Z(G)| = p^2$. Then prove that G is abelian. (Hint: what is $G/Z(G)$?)

5 pts (d) Derive contradiction from (c) and finish the proof that $Z(G)$ equals p^3 or p.

Answer.

(a) As $|Z(G)| = |G|$, $Z(G) = G$, i.e. G is abelian. By the fundamental theorem of finitely generated abelian groups, G is isomorphic to one of $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, or $\mathbb{Z}_{p^3}$.

(b) By Lagrange's theorem, $|Z(G)|$ is a divisor of $|G| = p^3$. On the other hand, by part (d) of the previous question (together with (b)) we see that $0 \equiv |G| \equiv |Z(G)| \pmod{p}$. Thus $|Z(G)| \neq 1$ and the result follows.

(c) $|G/Z(G)| = p$, thus $G/Z(G) \simeq \mathbb{Z}_p$. Let $a \in G$ be such that $\bar{a} \in G/Z(G)$ is a generator of $G/Z(G)$. Then any element in G is of the form $a^i c$ for some $i \in \mathbb{Z}$ and $c \in Z(G)$. Now for any $a^i c$ and $a^j d$ such that $i, j \in bZ$ and $c, d \in Z(G)$, we have $(a^i c)(a^j d) = a^{i+j} cd = (a^j d)(a^i c)$, thus G is abelian.

(d) If G is abelian then $G = Z(G)$, thus $|Z(G)| = p^3$. It contradicts the assumption that $|Z(G)| = p^2$ since $p \neq 1$. Thus $|Z(G)| = p$. In sum, $|Z(G)|$ should equal either p or p^3.