HOMEWORK 11 (DUE: 11:15 AM, DEC 7 FRI)

Remark. Note that the definition of rings in [Pin10] does *not* require that rings always have the unity. However, we will keep using our conventions, thus every ring should have the unity element. Also, similarly every ring homomorphism should send the unity to the unity.

1. Do Exercise 18.D.2–6 (except 1) in [Pin10].

2. If J is an ideal, then J is closed with respect to addition and J absorbs products in A by definition. For the converse, it suffices to show that J is an abelian group, i.e. closed under addition, $0 \in J$, and closed under taking (additive) inverse.

- J is closed under addition (this is given in the assumption.)
- Choose some $a \in J$, which is possible since J is nonempty by assumption. Since J absorbs products, $a \cdot 0 = 0 \in J$, thus J contains the additive identity.
- For any $a \in J$, we have $-a = (-1) \cdot a \in J$. (Note that $-1 \in A$ since A is a ring with unity.) Thus J is closed under taking additive inverse.

Therefore, J is an ideal of A.

3. Let I, J be ideals of A. We know (from the group theory) that $I \cap J$ is an abelian subgroup of A, thus it remains to show that $I \cap J$ absorbs products. But for any $x \in I \cap J$ and $a \in A$ we have $ax, xa \in I, ax, xa \in J$ since I and J are ideals. Thus we have $ax, xa \in I \cap J$ and the result follows.

4. For any $a \in A$, we have $a = a \cdot 1 \in J$. Thus $J = A$.

5. For any invertible element $u \in J$, we have $1 = uu^{-1} \in J$, thus $J = A$ by part 4.

6. If I is an ideal of F such that $I \neq \{0\}$, then I contains an invertible element, thus $I = F$ by part 5.

2. Do Exercise 18.E.1,2, and 4 (not all) in [Pin10].

1. We have $\phi(f+g) = (f+g)(0) = f(0)+g(0) = \phi(f)+\phi(g)$ and $\phi(fg) = (fg)(0) = f(0)g(0) = \phi(f)\phi(g)$. Also, $\phi(1) = 1$. (Here, $1 \in \mathcal{F}(\mathbb{R})$ is the constant function with value 1). Thus ϕ is a homomorphism. Moreover,

$$\ker \phi = \{f \in \mathcal{F}(\mathbb{R}) \mid f(0) = 0\} \text{ and } \operatorname{im} \phi = \mathbb{R},$$

since $\phi(c) = c$ for any $c \in \mathbb{R}$.

2. We have $h((a,b)+(c,d)) = h(a+c, b+d) = a+c = h(a,b)+h(c,d)$ and $h((a,b)(c,d)) = h(ac,bd) = ac = h(a,b)h(c,d)$. Also, $h(1,1) = 1$. Thus h is a homomorphism. Moreover,

$$\ker h = \{(a,b) \in \mathbb{R}\times\mathbb{R} \mid a = 0\} = \{0\}\times\mathbb{R} \text{ and } \operatorname{im} h = \mathbb{R},$$

since $h(x,0) = x$ for any $x \in \mathbb{R}$.

4. We have $h((a,b)+(c,d)) = h(a+c,b+d) = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = h(a,b)+h(c,d)$ and $h((a,b)(c,d)) = h(ac,bd) = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} = h(a,b)h(c,d)$. Also, $h(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus h is a homomorphism. Moreover,

$$\ker h = \{(a,b) \in \mathbb{R}\times\mathbb{R} \mid \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = 0\} = \{(0,0)\} \text{ and } \operatorname{im} h = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a,b \in \mathbb{R}\}.$$

3. Do Exercise 18.H.1–5 in [Pin10].

1. $jk \in J \cap K = \{0\}$, thus $jk = 0$.

2. $I_a = (a) + J + K$, thus it is an ideal.

3. (Here, the definition of $\operatorname{rad} J$ should be $\{a \in A \mid a^n \in J \text{ for some } n \in \mathbb{Z}_{>0}\}$.) First we check that $\operatorname{rad} J$ is an abelian group.

- For $a, b \in \operatorname{rad} J$, there exists $m, n \in \mathbb{Z}_{>0}$ such that $a^m, b^n \in J$. Thus
$$(a+b)^{m+n} = \sum_{i=0}^{m+n} \binom{m+n}{i} a^i b^{m+n-i} \in J$$
since either one of a^i or b^{m+n-i} is in J for any $0 \le i \le m+n$. But it means $a+b \in \operatorname{rad} J$.
- $0^1 = 0 \in J$, thus $0 \in \operatorname{rad} J$.
- If $a^n \in J$ for some $a \in A$ and $n \in \mathbb{Z}_{>0}$, then $(-a)^n = (-1)^n a^n \in J$, thus $-a \in \operatorname{rad} J$.

Thus J is an abelian group. Now for any $a \in A$ and $x \in \operatorname{rad} J$, there exists $n \in \mathbb{Z}_{>0}$ such that $x^n \in J$. Thus $(ax)^n = a^n x^n \in J$, which means that $ax \in \operatorname{rad} J$. Therefore, $\operatorname{rad} J$ is an ideal of A.

4. Let us write $\operatorname{Ann}(a) := \{x \in A \mid ax = 0\}$. We show that this is an ideal of A. First, we prove that it is an abelian group.

- For $x, y \in \operatorname{Ann}(a)$, $ax = ay = 0$, thus $a(x+y) = 0$ which means that $x + y \in \operatorname{Ann}(a)$.

- $a \cdot 0 = 0$, thus $0 \in \mathrm{Ann}(a)$.
- If $x \in \mathrm{Ann}(a)$, then $ax = 0$ thus $0 = -ax = a(-x)$ which means that $-x \in \mathrm{Ann}(a)$.

Thus $\mathrm{Ann}(a)$ is an abelian group. Now for any $x \in \mathrm{Ann}(a)$ and $y \in A$, we have $0 = ax = (ax)y = a(xy)$, thus $xy \in \mathrm{Ann}(a)$. Thus $\mathrm{Ann}(a)$ is an ideal of A.

This time we write $\mathrm{Ann}(A) := \{x \in A \mid ax = 0 \text{ for every } a \in A\}$. Similarly we first show that it is an abelian group.

- For $x, y \in \mathrm{Ann}(A)$, $ax = ay = 0$ for any $a \in A$, thus $a(x+y) = 0$ for any $a \in A$ which means that $x + y \in \mathrm{Ann}(A)$.
- $a \cdot 0 = 0$ for any $a \in A$, thus $0 \in \mathrm{Ann}(A)$.
- If $x \in \mathrm{Ann}(A)$, then $ax = 0$ for any $a \in A$ thus $0 = -ax = a(-x)$ which means that $-x \in \mathrm{Ann}(A)$.

Thus $\mathrm{Ann}(A)$ is an abelian group. Now for any $x \in \mathrm{Ann}(A)$ and $y \in A$, we have $0 = ax = (ax)y = a(xy)$ for any $a \in A$, thus $xy \in \mathrm{Ann}(A)$. Thus $\mathrm{Ann}(A)$ is an ideal of A. (Indeed, the fact that $\mathrm{Ann}(A)$ is an ideal follows from the statement that $\mathrm{Ann}(A) = \cap_{a\in A}\mathrm{Ann}(a)$, if we take for granted that an arbitrary intersection of ideals of A is again an ideal of A.)

If A is a ring with unity, then for any $x \in \mathrm{Ann}(A)$, we have $x = 1x = 0$. Thus $\mathrm{Ann}(A) = \{0\}$.

5. $\{0\}$ is clearly an abelian group and absorbs product, thus it is an ideal of A. Similar argument applies to A.

Now let $J = \{f \in \mathcal{F}(\mathbb{R}) \mid f(0) = 0\}$ and K be an ideal properly containing J, i.e. $J \subsetneq K$. Choose $g \in K - J$, then $g(x) - g(0) \in J$. Thus $g(0) = g(x) - (g(x) - g(0)) \in K$. Since $g \notin J$, $g(0) \neq 0$, thus $g(0)$ is invertible. But it means that $K = \mathcal{F}(\mathbb{R})$ (we proved in 18.D.5 that any ideal containing a unit is the whole ring.) It follows that J is maximal.

4. Do Exercise 18.I.1–2 (not all) in [Pin10].

1. $f(J)$ is an abelian subgroup of B by the group theory. Now for any $b \in B$ and $y \in f(J)$, there exists $a \in A$ and $x \in J$ such that $f(a) = b$ and $f(x) = y$. Thus

$$by = f(a)f(x) = f(ax) \in f(J), yb = f(x)f(a) = f(xa) \in f(J)$$

which means that $f(J)$ is an ideal of B.

2. The image $f(J)$ of any ideal J properly containing $\ker f$, i.e. $\ker f \subsetneq J$, is an ideal of B properly containing $\{0\}$ by part 1. But from 18.D.6 we know that B has only two ideals, $\{0\}$ and B. Thus $f(J) = B$. It means that for any

$a \in A$, there exists $x \in J$ such that $f(a) = f(x)$, i.e. $a - x \in \ker f$. But then $a \in x + \ker f \subset J$. Thus $J = A$. It follows that $\ker f$ is a maximal ideal of A.

References

[Pin10] Pinter, C. C., *A Book of Abstract Algebra*, 2nd ed., Dover Publications, 2010.