HOMEWORK 1 SOLUTIONS

1. Do Exercise 2.B.1–6. in [Pin10, pp. 23–24]. You do not need to justify your answers.

	Commutative	Associative	Identity	Inverses
1	N	N	N	—
2	N	N	N	—
3	Y	Y	N	—
4	Y	N	N	—
5	Y	N	N	—
6	Y	Y	N	—

2. Do Exercise 3.C.1–3. in [Pin10, pp. 30–31]. Here you *do* need to justify your answers.

1) For any $A \subset D$, we have

$$A + \emptyset = \emptyset + A = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A,$$

thus $\emptyset \in P_D$ is the identity element with respect to $+$.

2) For any $A \subset D$, we have

$$A + A = (A - A) \cup (A - A) = \emptyset,$$

thus A is the inverse of A with respect to $+$.

3) $P_D = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, D\}$. The operation table is given by

$+$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	D
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	D
$\{a\}$	$\{a\}$	$\emptyset$	$\{a,b\}$	$\{a,c\}$	$\{b\}$	$\{c\}$	D	$\{b,c\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\emptyset$	$\{b,c\}$	$\{a\}$	D	$\{c\}$	$\{a,c\}$
$\{c\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\emptyset$	D	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{b\}$	$\{a\}$	D	$\emptyset$	$\{b,c\}$	$\{a,c\}$	$\{a\}$
$\{a,c\}$	$\{a,c\}$	$\{c\}$	D	$\{a\}$	$\{b,c\}$	$\emptyset$	$\{a,b\}$	$\{b\}$
$\{b,c\}$	$\{b,c\}$	D	$\{c\}$	$\{b\}$	$\{a,c\}$	$\{a,b\}$	$\emptyset$	$\{a\}$
D	D	$\{b,c\}$	$\{a,c\}$	$\{a,b\}$	$\{c\}$	$\{b\}$	$\{a\}$	$\emptyset$

3. [Fra02, Exercise 4.19] Let $S := \mathbb{R} - \{-1\}$ with the operation

$$* : S \times S \to S : (a, b) \mapsto a + b + ab.$$

a. Show that $(S, *)$ is a group.

It suffices to show the following properties.

- $*$ is associative. For any $a, b, c \in S$, we have
$$\begin{aligned}(a * b) * c &= (a + b + ab) * c \\ &= a + b + c + ab + ac + bc + abc \\ &= a * (b + c + bc) \\ &= a * (b * c),\end{aligned}$$
from which the associativity follows.
- The identity element exists. For any $a \in S$, we have
$$a * 0 = 0 * a = a,$$
thus $0 \in S$ is the identity element.
- Every element has its own inverse. For any $a \in S$, we have
$$a * \frac{-a}{a+1} = \frac{-a}{a+1} * a = a + \frac{-a}{a+1} + \frac{-a^2}{a+1} = 0,$$
thus $\frac{-a}{a+1} \in S$ is the inverse of a. Here, as $a \in S = \mathbb{R} - \{-1\}, \frac{-a}{a+1} \in \mathbb{R}$ is well-defined. Furthermore, $\frac{-a}{a+1} = -1$ if and only if $a = a + 1$ which is impossible, thus $\frac{-a}{a+1} \neq -1$ and we have $\frac{-a}{a+1} \in S = \mathbb{R} - \{-1\}$ as well.

b. Find the solution of the equation $2 * x * 3 = 7$ in S.

We have
$$7 = 2 * x * 3 = 2 + x + 3 + 2x + 3x + 6 + 6x = 11 + 12x.$$
Thus $x = -1/3$.

4. [Sar08, Exercise 3.11] Let $(G, *)$ be a group such that $x^2 = e$ for all $x \in G$. Show that $(G, *)$ is abelian, i.e. $*$ is commutative.

For any $x, y \in G$, it suffices to show that $xy = yx$, or equivalently $xyx^{-1}y^{-1} = e$. But since $x^2 = e$ for all $x \in G$, $x = x^{-1}$ for all $x \in G$ and we have
$$xyx^{-1}y^{-1} = xyxy = (xy)^2 = e$$
as desired.

5. [Sar08, Exercise 3.14] Let G be a nonempty set with an associative operation $* : G \times G \to G$. If $\forall a, b \in G, \exists x, y \in G$ such that $a * x = y * a = b$, then show that $(G, *)$ is a group.

It suffices to check the following properties of $(G, *)$.

- $*$ is associative. It is already given from the assumption.
- The identity element exists in G. Since G is nonempty, there exists an element $a \in G$. Then by the assumption there also exist $e_r, e_l \in G$ such that $ae_r = e_l a = a$. We claim that for such e_r, e_l we have $be_r = e_l b = b$ for any $b \in G$; indeed, for $b \in G$ let $x, y \in G$ be such that $ax = ya = b$. Then we have

  $$b = ya = yae_r = be_r \quad \text{and} \quad b = ax = e_l ax = e_l b,$$

  which is what we claimed above. In particular, we also have

  $$e_l = e_l e_r = e_r.$$

  For simplicity we denote $e_l = e_r$ by e. Then as $be = eb = b$ for any $b \in G$, we see that $e \in G$ is the identity element with respect to $*$.
- Every element has its own inverse. For $a \in G$, by assumption there exists $x, y \in G$ such that $ax = ya = e$. But we have

  $$x = ex = yax = ye = y,$$

  thus $x = y$ is the inverse of a.

References

[Fra02] Fraleigh, J. B., *A First Course in Abstract Algebra*, 7th ed., Pearson, 2002.
[Pin10] Pinter, C. C., *A Book of Abstract Algebra*, 2nd ed., Dover Publications, 2010.
[Sar08] Saracino, D., *Abstract Algebra: A First Course*, 2nd ed., Waveland Press, 2008.