HOMEWORK 2 (DUE: 11:15 AM, SEP 26 WED)

1. Do Exercise 4.H.1–8. in [Pin10, p. 43].

1) It is obvious when $n = 1$. In general if $(bab^{-1})^k = ba^kb^{-1}$ then
$$(bab^{-1})^{k+1} = (bab^{-1})^k(bab^{-1}) = ba^kb^{-1}bab^{-1} = ba^kab^{-1} = ba^{k+1}b^{-1},$$
thus it is true for $n = k+1$. Now the result follows from induction on n.

2) It is obvious when $n = 1$. In general if $(ab)^k = a^kb^k$ then
$$\begin{aligned}(ab)^{k+1} &= (ab)^k(ab) = a^kb^kab = a^k(b^{k-1}ab)b = a^k(b^{k-2}ab^2)b = \cdots \\ &= a^k(ab^k)b = a^{k+1}b^{k+1},\end{aligned}$$
thus it is true for $n = k+1$. Now the result follows from induction on n.

3) Since $xax = e$, we have $(xa)^2 = xaxa = a$. Thus $(xa)^{2n} = ((xa)^2)^n = a^n$.

4) $(a^2)^2 = a^4 = a$, thus a has a square root a^2.

5) $a^3 = a$, thus a has a cube root a.

6) There exists $b \in G$ such that $b^3 = a^{-1}$ by assumption. It implies $a = (b^3)^{-1} = (b^{-1})^3$, thus a has a square root (which is b^{-1}).

7) We have $(xax)^3 = xa(x^2ax)xax = xaa^{-1}xax = x^2ax = a^{-1}$, thus a^{-1} has a cube root. Now the result follows from part 6).

8) Since $xax = b$, $(ax)^2 = axax = ab$. Thus ab has a square root ax.

2. Do Exercise 5.B.1–6. in [Pin10, p. 49].

In each case we need to show that (a) H is closed under $+$, (b) H contains the identity, (c) H is closed under taking inverse, i.e. $f \mapsto -f$. Note that the identity of G is the zero function $\zeta : \mathbb{R} \to \mathbb{R} : x \mapsto 0$. From now on we assume that $f, g \in H$ and show that $f+g, -f, \zeta \in H$.

1) For $x \in [0,1]$,
$$(f+g)(x) = f(x) + g(x) = 0, \quad (-f)(x) = -f(x) = 0, \quad \zeta(x) = 0.$$
Thus $f+g, -f, \zeta \in H$.

2) For $x \in \mathbb{R}$,
$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x),$$
$$(-f)(-x) = -f(-x) = f(x) = -(-f)(x), \quad \zeta(-x) = 0 = -\zeta(x).$$
Thus $f+g, -f, \zeta \in H$.

3) For $x \in \mathbb{R}$ and $n \in \mathbb{Z}$,
$$(f+g)(x+n\pi) = f(x+n\pi) + g(x+n\pi) = f(x) + g(x) = (f+g)(x),$$
$$(-f)(x+n\pi) = -f(x+n\pi) = -f(x) = (-f)(x), \quad \zeta(x+n\pi) = 0 = \zeta(x).$$
Thus $f+g, -f, \zeta \in H$.

4) We have
$$\int_0^1 (f+g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = 0,$$
$$\int_0^1 (-f)(x)dx = -\int_0^1 f(x)dx = 0, \quad \int_0^1 \zeta(x)dx = \int_0^1 0dx = 0.$$
Thus $f+g, -f, \zeta \in H$.

5) Let $c, d \in \mathbb{R}$ be constants such that $df/dx = c$ and $dg/dx = d$. Then,
$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} = c + d, \quad \frac{d(-f)}{dx} = -\frac{df}{dx} = -c, \quad \frac{d\zeta}{dx} = 0$$
and $c+d, -c, 0$ are constants. Thus $f+g, -f, \zeta \in H$.

6) For any $x \in \mathbb{R}$,
$$(f+g)(x) = f(x) + g(x) \in \mathbb{Z}, \quad (-f)(x) = -f(x) \in \mathbb{Z}, \quad \zeta(x) = 0 \in \mathbb{Z}$$
since $(\mathbb{Z}, +)$ is a group. Thus $f+g, -f, \zeta \in H$.

3. Do Exercise 5.C.1–7. in [Pin10, p. 49]. For 5.C.7, you do not need to prove it – just explain why you think that 5.C.4–6 may not be true for nonabelian groups. (Of course you can give a counterexample to each statement, but it is not required.)

1) For $x, y \in H$, $(xy)(xy) = xyxy = x^2y^2 = e$, thus $xy = (xy)^{-1}$ which implies $xy \in H$. Also $(x^{-1})^{-1} = x = x^{-1}$, thus $x^{-1} \in H$. Finally $e^2 = e$, i.e. $e = e^{-1}$, thus $e \in H$. Thus H is a subgroup of G.

2) For $x, y \in H$, $(xy)^n = x^n y^n = e$ and $(x^{-1})^n = (x^n)^{-1} = e$, thus $xy, x^{-1} \in H$. Finally $e^n = e$, thus $e \in H$. Thus H is a subgroup of G.

3) For $x, y \in H$, let $a, b \in G$ be such that $a^2 = x$ and $b^2 = y$. (Such a and b exist by assumption.) Then $(ab)^2 = abab = a^2b^2 = xy$ and $(a^{-1})^2 = (a^2)^{-1} = x^{-1}$, thus $xy, x^{-1} \in H$. Finally $e^2 = e$, thus $e \in H$. Thus H is a subgroup of G.

4) For $x, y \in K$, $(xy)^2 = xyxy = x^2y^2 \in H$ and $(x^{-1})^2 = (x^2)^{-1} \in H$ since H is a subgroup of G. Thus $xy, x^{-1} \in K$. Finally $e^2 = e \in H$, thus $e \in K$. Thus K is a subgroup of G.

5) For $x, y \in K$, there exists $m, n \in \mathbb{Z}_{>0}$ such that $x^m, y^n \in H$. Now

$$(xy)^{mn} = x^{mn}y^{mn} = (x^m)^n(y^n)^m \in H, \quad (x^{-1})^m = (x^m)^{-1} \in H, \quad e^1 = e \in H,$$

thus $xy, x^{-1}, e \in K$. Therefore K is a subgroup of G.

6) For $x, z \in H$ and $y, w \in K$, we have

$$(xy)(zw) = (xz)(yw) \in HK, \quad (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} \in HK, \quad e = ee \in HK.$$

Thus HK is a subgroup of G.

7) The problem is that in the proof of 4) – 6) we used properties which is true when G is abelian. Here we give counterexamples to those with assuming that G is abelian.

   4) Let $G = S_3$ and $H = \{id\} \subset G$. Then $K = \{id, (12), (23), (13)\}$. However, K is not a subgroup of G; in fact K generates G as we proved in the class.

   5) Let $\mathrm{Aut}(\mathbb{Z})$ be the permutation group of $\mathbb{Z}$, i.e. the group of all bijective functions from $\mathbb{Z}$ to $\mathbb{Z}$ and define

   $$\alpha : \mathbb{Z} \to \mathbb{Z} : x \mapsto -x, \quad \beta : \mathbb{Z} \to \mathbb{Z} : x \mapsto 1 - x.$$

   Let $G \subset \mathrm{Aut}(\mathbb{Z})$ be the subgroup of $\mathrm{Aut}(\mathbb{Z})$ generated by α and β, i.e. $G = \langle \alpha, \beta \rangle$. Also let $H = \{id\} \subset G$. Then

   $$K = \{x \in G \mid x^n = id \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

   Since $\alpha^2 = \beta^2 = id$, we have $\alpha, \beta \in K$. Thus if K is a subgroup of G, then $\alpha \circ \beta \in K$. However,

   $$\alpha \circ \beta : \mathbb{Z} \to \mathbb{Z} : x \mapsto x - 1$$

   is clearly not in K; $(\alpha \circ \beta)^n(0) = -n \neq 0$, thus $(\alpha \circ \beta)^n \neq id$ for any $n \in \mathbb{Z}_{>0}$.

   6) Let $G = S_3$, $H = \{id, (12)\}$, $K = \{id, (23)\}$. Then

   $$HK = \{id, (12), (23), (12)(23) = (123)\}.$$

   But it is clearly not a subgroup of G.

4. [Fra02, Exercise 7.1–6] In each case, describe the subgroup generated by the given subset. Here the operation on each group is given by $+$. (You do not need to justify your answers.)

   a) $\{2, 3\} \subset \mathbb{Z}_{12}$

   $\langle\{2, 3\}\rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = \mathbb{Z}_{12}$

b) $\{4, 6\} \subset \mathbb{Z}_{12}$

$\langle\{4, 6\}\rangle = \{0, 2, 4, 6, 8, 10\}$

c) $\{8, 10\} \subset \mathbb{Z}_{18}$

$\langle\{8, 10\}\rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$

d) $\{12, 30\} \subset \mathbb{Z}_{36}$

$\langle\{12, 30\}\rangle = \{0, 6, 12, 18, 24, 30\}$

e) $\{12, 42\} \subset \mathbb{Z}$

$\langle\{12, 42\}\rangle = \{\ldots, -12, -6, 0, 6, 12, \ldots\} = 6\mathbb{Z}$

f) $\{18, 24, 39\} \subset \mathbb{Z}$

$\langle\{18, 24, 39\}\rangle = \{\ldots, -6, -3, 0, 3, 6, \ldots\} = 3\mathbb{Z}$

5. [Sar08, Exercise 5.18.a)] Show that it is impossible for a group G to be the union of two *proper* subgroups, i.e. if there exist two subgroups $H, K \subset G$ such that $H \cup K = G$, then either $H = G$ or $K = G$.

Suppose otherwise, i.e. $H \cup K = G$ but $H, K \subsetneq G$. Choose $h \in H - K, k \in K - H$. Then $hk \in G$, thus either $hk \in H$ or $hk \in K$. If $hk \in H$, then $k = h^{-1}(hk) \in H$, which contradicts our assumption. Similarly if $hk \in K$, then $h = (hk)k^{-1} \in K$ which is again a contradiction. Thus we should have $H = G$ or $K = G$ as desired.

References

[Fra02] Fraleigh, J. B., *A First Course in Abstract Algebra*, 7th ed., Pearson, 2002.
[Pin10] Pinter, C. C., *A Book of Abstract Algebra*, 2nd ed., Dover Publications, 2010.
[Sar08] Saracino, D., *Abstract Algebra: A First Course*, 2nd ed., Waveland Press, 2008.