HOMEWORK 4 (DUE: 11:15 AM, OCT 10 WED)

1. Do Exercise 9.G.1–4. in [Pin10, p. 100].

9.G.1. Since for any $x, y \in \mathbb{R}^*$

$$\begin{aligned} f(x) * f(y) &= (x-1) * (y-1) = (x-1) + (y-1) + (xy - x - y + 1) \\ &= xy - 1 = f(xy), \end{aligned}$$

$f : \mathbb{R}^* \to G$ is a homomorphism. But it is clearly a bijection with the inverse $f^{-1}(x) = x + 1$. Thus f is an isomorphism.

9.G.2. Define $f : \mathbb{R} \to G : x \mapsto x - 1$. Then it is clearly a bijection. Furthermore,

$$\begin{aligned} f(x+y) &= x + y - 1 = (x-1) + (y-1) + 1 \\ &= (x-1) * (y-1) = f(x) * f(y), \end{aligned}$$

thus f is a homomorphism. Therefore, f is an isomorphism.

9.G.3. Set $f : \mathbb{R}^* \to G : x \mapsto 2x$. Then it is clearly a bijection. Furthermore,

$$f(xy) = 2xy = \frac{(2x)(2y)}{2} = (2x) * (2y) = f(x) * f(y),$$

thus f is a homomorphism. Therefore, f is an isomorphism.

9.G.4. If we define $g : \mathbb{R}^* \to \mathbb{R}^{\text{pos}} \times \mathbb{Z}_2 : x \mapsto (|x|, \varepsilon(x))$ where $\varepsilon(x) = 0$ if $x > 0$ and $\varepsilon(x) = 1$ if $x < 0$, then f and g are inverse to each other, thus f is a bijection. Furthermore, we have

$$\begin{aligned} f((x,y)(z,w)) &= f(xz, y+w) = (-1)^{y+w} xz \\ &= ((-1)^y x)((-1)^w z) = f(x,y) f(z,w), \end{aligned}$$

which means that f is a homomorphism. Thus f is an isomorphism.

2. Do Exercise 14.B.1–6. in [Pin10, pp. 142–143].

14.B.1. We have $\phi(f+g) = (f+g)(0) = f(0) + g(0) = \phi(f) + \phi(g)$, thus f is a homomorphism. Also,

$$\ker \phi = \{f \in \mathscr{F}(\mathbb{R}) \mid f(0) = 0\}.$$

14.B.2. We have $\phi(f+g) = (f+g)' = f' + g' = \phi(f) + \phi(g)$, thus f is a homomorphism. Also,

$$\ker \phi = \{f \in \mathscr{D}(\mathbb{R}) \mid f' = 0\} = \{\text{constant functions on } \mathbb{R}\}.$$

14.B.3. We have $f((x,y)+(z,w)) = f(x+z, y+w) = x+y+z+w = f(x,y)+f(z,w)$, thus f is a homomorphism. Also,

$$\ker f = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x+y=0\} = \{(x,-x) \in \mathbb{R}\times\mathbb{R} \mid x \in \mathbb{R}\}.$$

14.B.4. We have $f(xy) = |xy| = |x||y| = f(x)f(y)$, thus f is a homomorphism. Also,

$$\ker f = \{x \in \mathbb{R}^* \mid |x| = 1\} = \{1, -1\}.$$

14.B.5. We have

$$\begin{aligned} f((a+b\mathbf{i})(c+d\mathbf{i})) &= f((ac-bd)+(bc+ad)\mathbf{i}) = \sqrt{(ac-bd)^2+(bc+ad)^2} \\ &= \sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2} = \sqrt{(a^2+b^2)(c^2+d^2)} \\ &\sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)} = f(a+b\mathbf{i})f(c+d\mathbf{i}), \end{aligned}$$

thus f is a homomorphism. Also,

$$\ker f = \{a+b\mathbf{i} \in \mathbb{C}^* \mid \sqrt{a^2+b^2} = 1\} = (\text{the unit circle in } \mathbb{C}^*).$$

14.B.6. We have

$$\begin{aligned} f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) &= f\left(\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\right) \\ &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= acef + bcfg + adeh + bdgh - acef - bceh - adfg - bdgh \\ &= bcfg + adeh - bceh - adfg \\ &= (ad-bc)(eh-fg) \\ &= f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\left(\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right), \end{aligned}$$

thus f is a homomorphism. Also,

$$\ker f = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid ad - bc = 1\right\}.$$

(It is called a special linear group, denoted SL_2.)

3. Do Exercise 14.G.1–6. in [Pin10, pp. 144–145].

14.G.1. For any $x, y \in H$, there exists $a, b \in G$ such that $f(a) = x$ and $f(b) = y$. Thus,

$$xy = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = yx,$$

which means that H is abelian.

14.G.2. Let $g \in G$ be a generator of G. Then for any $h \in H$, there exists $a \in G$ such that $f(a) = h$ and $a = g^m$ for some $m \in \mathbb{Z}$. In particular, we have $h = f(a) = f(g^m) = f(g)^m$. Thus H is a cyclic group generated by $f(g)$.

14.G.3. For any $h \in G$, there exists $g \in G$ such that $f(g) = h$. If $m \in \mathbb{Z}_{>0}$ is an integer such that $g^m = e$, then $h^m = f(g)^m = f(g^m) = f(e) = e$. Thus the order of every element in H is finite.

14.G.4. For $h \in H$, there exists $g \in G$ such that $f(g) = h$. Then $h = f(g) = f(g^{-1}) = f(g)^{-1} = h$, thus every element in H is its own inverse.

14.G.5. For $h \in H$, there exists $g \in G$ such that $f(g) = h$. Choose $x \in G$ such that $x^2 = g$ which exists by assumption. Then $f(x)^2 = f(x^2) = f(g) = h$, thus $f(x)$ is a square root of h. It follows that every element in H has a square root.

14.G.6. Suppose that G is generated by $g_1, g_2, \ldots, g_n$. Then for any $h \in H$, there exists $g \in G$ such that $f(g) = h$ and also we may write $g = g_{i_1}^{m_1} g_{i_2}^{m_2} \cdots g_{i_k}^{m_k}$ for some $g_{i_1}, g_{i_2}, \ldots, g_{i_k} \in \{g_1, g_2, \ldots, g_n\}$ and $m_1, m_2, \ldots, m_k \in \mathbb{Z}$. But then we have

$$h = f(g) = f(g_{i_1}^{m_1} g_{i_2}^{m_2} \cdots g_{i_k}^{m_k}) = f(g_{i_1})^{m_1} f(g_{i_2})^{m_2} \cdots f(g_{i_k})^{m_k}$$

and $f(g_{i_1}), f(g_{i_2}), \ldots, f(g_{i_k}) \in \{f(g_1), f(g_2), \ldots, f(g_n)\}$. Thus H is generated by $\{f(g_1), f(g_2), \ldots, f(g_n)\}$ and the claim follows.

References

[Pin10] Pinter, C. C., *A Book of Abstract Algebra*, 2nd ed., Dover Publications, 2010.