HOMEWORK 6 (DUE: 11:15 AM, OCT 24 WED)

1. Do Exercise 15.C.1–3. (*not all*) in [Pin10, p. 153].

> 15.C.1. $x^2 \in H \Leftrightarrow (xH)^2 = x^2H = H \in G/H \Leftrightarrow xH = (xH)^{-1} \in G/H$. Thus $x^2 \in H$ for every $x \in G$ if and only if xH is its own inverse for every $xH \in G/H$.
>
> 15.C.2. $x^m \in H \Leftrightarrow (xH)^m = x^mH = H \in G/H \Leftrightarrow \mathrm{ord}(xH)|m$. Thus $x^m \in H$ for every $x \in G$ if and only if the order of xH divides m for every $xH \in G/H$.
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> 15.C.3. (Here the exercise is falsely stated; "an integer n" should be "a positive integer n". Here we prove the statement after this modification.)
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> If for every x there exists a positive integer n such that $x^n \in H$, then $x^n \in H \Leftrightarrow (xH)^n = x^nH = H \in G/H$ thus $\mathrm{ord}(xH)$ is finite. Conversely, if every element of G/H has finite order then for any $x \in G$ we have $x^{\mathrm{ord}(xH)} \in H$.

2. Do Exercise 15.E.1–3. (*not all*) in [Pin10, p. 154].

> 15.E.1. Since $a^{\mathrm{ord}(a)} = e$, $(Ha)^{\mathrm{ord}(a)} = He$. Thus $\mathrm{ord}(Ha)|\mathrm{ord}(a)$.
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> 15.E.2. This follows from Lagrange's theorem and that $|G/H| = (G : H) = m$.
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> 15.E.3. By part 1 and 2, we have $\mathrm{ord}(Ha)|p$ and $\mathrm{ord}(Ha)|\mathrm{ord}(a)$. Since $a \notin H$, $\mathrm{ord}(Ha) \neq 1$, thus $\mathrm{ord}(Ha) = p$ because p is a prime. Thus $p|\mathrm{ord}(a)$.

3. Do Exercise 15.F.1–4. in [Pin10, p. 154].

> 15.F.1. It follows from the assumption that $\langle Ca \rangle = G/C$.
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> 15.F.2. It directly follows from part 1.
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> 15.F.3. Let us write $x, y \in G$ to be $x = ca^m, y = c'a^n$ for some $c, c' \in C$. Then,
>
> $$xy = ca^mc'a^n = cc'a^{m+n} = c'a^nca^m = yx.$$
>
> Thus $xy = yx$.
>
> 15.F.4. It follows from part 3.

4. Do Exercise 15.G.1–4. (*not all*) in [Pin10, pp. 154–155].

15.G.1. Let us write $[a]$ to be the conjugacy class of a. Then,

$$[a] = \{a\} \Leftrightarrow \{gag^{-1} \mid g \in G\} = \{a\} \Leftrightarrow gag^{-1} = a \ \forall g \in G \Leftrightarrow a \in C.$$

15.G.2. After reordering $k_1, k_2, \ldots, k_t$, we may assume that $k_1, k_2, \ldots, k_{s-1}$ are the sizes of conjugacy classes in C. By part 1, we have $k_1 = k_2 = \cdots = k_{s-1}$ and also $s - 1 = |C| = c$. Thus we have $|G| = c + k_s + k_{s+1} + \cdots + k_t$ as desired.

15.G.3. We need to show that the size of $[x]$ is a power of p if $x \notin C$. But we know that $|[x]| = (G : C_x) = |G|/|C_x|$ where C_x is the centralizer of x in G. In particular, $|[x]| \mid |G| = p^k$. Thus $|[x]|$ is a power of p.

15.G.4. We know that $k_i > 1$ for $i \in \{s, s+1, \ldots, t\}$ by part 1. Thus by part 3, k_i are multiple of p. Now if we divide both sides of $|G| = c + k_s + k_{s+1} + \cdots + k_t$ by p, then we have $c \equiv 0 \mod p$, from which the result follows.

References

[Pin10] Pinter, C. C., *A Book of Abstract Algebra*, 2nd ed., Dover Publications, 2010.