HOMEWORK 7 (BONUS!) (DUE: 11:15 AM, OCT 31 WED)

1. [Sar08, Exercise 14.1] and [Fra02, Exercise 11.21–25] Find, up to isomorphism, all abelian groups of order:

   (a) 32 (b) 48 (c) 72 (d) 84 (e) 450 (f) 720 (g) 1089

   Also, find the maximum possible order for some element in each group. You do not need to justify your answer.

   For brevity, we write $(a_1, a_2, \ldots, a_r)$ instead of $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \cdots \mathbb{Z}_{a_r}$. Thus for example we have $(3, 4) = (12)$. Also we write $[(a_1, a_2, \ldots, a_r), n]$ if the maximum possible order for some element in $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \cdots \mathbb{Z}_{a_r}$ is n.

   (a) $[(2,2,2,2,2),2]$, $[(2,2,2,4),4]$, $[(2,4,4),4]$, $[(2,2,8),8]$, $[(4,8),8]$, $[(2,16),16]$, $[(32),32]$

   (b) $[(2,2,2,2,3) = (2,2,2,6),6]$, $[(2,2,4,3) = (2,2,12),12]$, $[(4,4,3) = (4,12),12]$, $[(2,8,3) = (2,24),24]$, $[(16,3) = (48),48]$

   (c) $[(2,2,2,3,3) = (2,6,6),6]$, $[(2,2,2,9) = (2,2,18),18]$, $[(2,4,3,3) = (6,12),12]$, $[(2,4,9) = (2,36),36]$, $[(8,3,3) = (3,24),24]$, $[(8,9) = (72),72]$

   (d) $[(2,2,3,7) = (2,42),42]$, $[(4,3,7) = (84),84]$

   (e) $[(2,3,3,5,5) = (15,30),30]$, $[(2,3,3,25) = (3,150),150]$, $[(2,9,5,5) = (5,90),90]$, $[(2,9,25) = (450),450]$

   (f) $[(2,2,2,2,3,3,5) = (2,2,6,30),30]$, $[(2,2,2,2,9,5) = (2,2,2,90),90]$, $[(2,2,4,3,3,5) = (2,6,60),60]$, $[(2,2,4,9,5) = (2,2,180),180]$, $[(4,4,3,3,5) = (12,60),60]$, $[(4,4,9,5) = (4,180),180]$, $[(2,8,3,3,5) = (6,120),120]$, $[(2,8,9,5) = (2,360),360]$, $[(16,3,3,5) = (3,240),240]$, $[(16,9,5) = (720),720]$

   (g) $[(3,3,11,11) = (33,33),33]$, $[(3,3,121) = (3,363),363]$, $[(9,11,11) = (11,99),99]$, $[(9,121) = (1089),1089]$

2. [Fra02, 11.29] Here you do not need to justify your answer.

(a) Let p be a prime number. Find the number of abelian groups, up to isomorphism, of order p^n when $n = 2, 3, 4, 5, 6, 7, 8$.

> Answer: 2, 3, 5, 7, 11, 15, 22

(b) Let p, q, r be distinct prime numbers. Find the number of abelian groups, up to isomorphism, of order:

i. $p^3q^4r^7$ ii. $(qr)^7$ iii. $p^5q^4r^3$.

> i. $3 \cdot 5 \cdot 15 = 225$ ii. $15 \cdot 15 = 225$ iii. $7 \cdot 5 \cdot 3 = 105$

3. [Fra02, Exercise 11.10] Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. You do not need to justify your answer.

$$\{(0,0,0)\}, \quad \{(0,0,0),(1,0,0)\}, \{(0,0,0),(0,1,0)\}, \{(0,0,0),(0,0,1)\},$$
$$\{(0,0,0),(1,1,0)\}, \{(0,0,0),(1,0,1)\}, \{(0,0,0),(0,1,1)\}, \{(0,0,0),(1,1,1)\},$$
$$\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}, \{(0,0,0),(1,0,0),(0,0,1),(1,0,1)\},$$
$$\{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}, \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$$
$$\{(0,0,0),(1,1,0),(0,0,1),(1,1,1)\}, \{(0,0,0),(1,0,1),(0,1,0),(1,1,1)\}$$
$$\{(0,0,0),(0,1,1),(1,0,0),(1,1,1)\}, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

4. [Fra02, Exercise 11.15–20] Here you *need* to justify your answer.

(a) Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not?

> They are isomorphic since $\mathbb{Z}_2 \times \mathbb{Z}_{12} \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \simeq \mathbb{Z}_4 \times \mathbb{Z}_6$.

(b) Are the groups $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ isomorphic? Why or why not?

> They are not isomorphic since
>
> $$\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24} \simeq \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5,$$
> $$\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40} \simeq \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$
>
> and these two expressions are different.

(c) Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Why or why not?

> They are isomorphic since they are both isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5$.

References

[Fra02] Fraleigh, J. B., *A First Course in Abstract Algebra*, 7th ed., Pearson, 2002.
[Sar08] Saracino, D., *Abstract Algebra: A First Course*, 2nd ed., Waveland Press, 2008.