HOMEWORK 8 (DUE: 11:15 AM, NOV 7 WED)

1. [Fra02, Exercise 16.12] Let X be a G-set and let $Y \subset X$. Let

$$G_Y := \{g \in G \mid gy = y \text{ for all } y \in Y\}.$$

Show that G_Y is a subgroup of G.

> - For $g, h \in G_Y$, $gy = y = hy$ for all $y \in Y$. Thus $(gh)y = g(hy) = gy = y$ which means $gh \in G_Y$.
> - $e \in G$ acts by identity on X, thus $ey = y$ for all $y \in Y$, i.e. $e \in G_Y$.
> - If $g \in G_Y$ then $gy = y$ for all $y \in Y$, thus $g^{-1}y = (g^{-1}g)y = ey = y$. It follows that $g^{-1} \in G_Y$.
>
> Thus G_Y is a subgroup of G.

2. [Fra02, Exercise 16.13] Suppose that $G = \mathbb{R}$ acts on $\mathbb{R}^2$ in a way that $x \in \mathbb{R}$ acts by the counterclockwise rotation of $\mathbb{R}^2$ about the origin through x radians.

(a) Show that this action is well-defined.

> It is equivalent to saying that for $x, y \in \mathbb{R}$, (rotation by x radians)$\circ$(rotation by y radians) = (rotation by $x + y$ radians). But it is clear from the definition of rotation.

(b) Describe each orbit of $\mathbb{R}$ in $\mathbb{R}^2$ geometrically.

> It is clear that any two points $p, q \in \mathbb{R}^2$ are in the same orbit then the distance from the origin to p and q are the same. Conversely, if p and q are of the same distance from the origin then one can find a rotation which moves p to q. Therefore, each orbit is a circle centered at origin with some radius $r \geq 0$. (In particular when $r = 0$, then the origin itself is an orbit.)

(c) For each $p \in \mathbb{R}^2$, what is $\text{Stab}_{\mathbb{R}}(p)$?

> If $0 \neq p \in \mathbb{R}^2$, then p is fixed by rotation if and only if the rotation is trivial. It is equivalent to that x is a multiple of 2π, i.e. $x = 2\pi n$ for some $n \in \mathbb{Z}$. Thus $\text{Stab}_{\mathbb{R}}(p) = \langle 2\pi \rangle \subset \mathbb{R}$. On the other hand if p is the origin, then any rotation fixes p. Thus $\text{Stab}_{\mathbb{R}}(p) = \mathbb{R}$.

3. The edges of a regular n-gon are to be painted with r different colors. If it is allowed to use each color several times, find the number of ways for such paintings up to rotation if
   (a) $(n, r) = (6, 2)$,
   (b) $(n, r) = (3, 4)$.

   Briefly justify your answers.

Here we use Burnside's lemma. Let X be the set of all possible paintings of edges of a regular n-gon and G be a group of rotations isomorphic to $\mathbb{Z}_n$. We write $G = \{e, \rho, \rho^2, \ldots, \rho^{n-1}\}$ where ρ is the counterclockwise rotation by $\frac{360}{n}$ degrees.

(a) $(n, r) = (6, 2)$. Let us label colors of the edges by a, b, c, d, e, f in a counterclockwise order. There are $r = 2$ possible choices for each color, thus $|X| = 2^6 = 64$, which implies $|X^e| = |X| = 64$. Now any painting in X^ρ should satisfy the condition $a = b = c = d = e = f$, thus if we choose the color of a then it is completely determined. It means that $|X^\rho| = 2^1 = 2$ and also $|X^{\rho^5}| = |X^\rho| = 2$ since $\rho^{-1} = \rho^5$. Any painting in X^{ρ^2} should satisfy the condition $a = c = e, b = d = f$, thus the choice for a and b determines the painting. It implies that $|X^{\rho^2}| = 2^2 = 4$ and similarly $|X^{\rho^4}| = |X^{\rho^2}| = 4$ since $(\rho^2)^{-1} = \rho^4$. Finally, any painting in X^{ρ^3} is characterized by the condition $a = d, b = e, c = f$, thus there are three colors to choose. It means that $|X^{\rho^3}| = 2^3 = 8$. Now we use Burnside's lemma to conclude that

$$\#(G - \text{orbits in } X) = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14.$$

(b) $(n, r) = (3, 4)$.

We argue similarly to above. This time $G = \{e, \rho, \rho^2\}$ and direct calculation shows that $|X^e| = |X| = 4^3 = 64, |X^\rho| = |X^{\rho^2}| = 4$. Thus,

$$\#(G - \text{orbits in } X) = \frac{1}{3}(64 + 4 + 4) = 24.$$

4. Do the problem 3 if "up to rotation" is replaced by "up to rotation and reflection".

The only change is that we need to replace $G \simeq \mathbb{Z}_n$ by $G = D_n$, which includes n reflections. Let us write $G = \{e, \rho, \ldots, \rho^{n-1}, r_1, r_2, \ldots, r_n\}$ where $r_1, r_2, \ldots, r_n$ are the reflections which fix the regular n-gon.

(a) $(n, r) = (6, 2)$. There are two kinds of reflections we need to consider. Suppose that r is a reflection whose axis is perpendicular to a pair of edges of the hexagon.

Without loss of generality we may assume that the colors of these edges are labeled a and d. Then paintings in X^r are characterized by the condition $b = f, c = e$. Thus the choice for a, b, c, d determines each coloring in X^r, which means $|X^r| = 2^4 = 16$. On the other hand, if the axis of such a reflection r passes through two opposite vertices of the regular hexagon, then r has three orbits in the set of edges of the hexagon, which means that $|X^r| = 2^3 = 8$. Note that there are 3 reflections of the former kind and also 3 reflections of the latter kind. Thus we conclude that

$$\#(G-\text{orbits in } X) = \frac{1}{12}(64+2+4+8+4+2+16+16+16+8+8+8) = 13.$$

(b) $(n, r) = (3, 4)$.

We argue similarly to above. This time $G = \{e, \rho, \rho^2, r_1, r_2, r_3\}$. Also for any reflection r, we have $|X^r| = 4^2 = 16$. Thus

$$\#(G-\text{orbits in } X) = \frac{1}{6}(64+4+4+16+16+16) = 20.$$

References

[Fra02] Fraleigh, J. B., *A First Course in Abstract Algebra*, 7th ed., Pearson, 2002.