PROBLEM SET 5 (DUE: OCT 16, WED)

1. Do Exercise 3.1.5. in [OS18]. (You may use a computer program to draw unit circles.)
   
   (a) \(x^2 + y^2 = 1\)  
   (b) \(2x^2 + 5y^2 = 1\)  
   (c) \(x^2 - 2xy + 4y^2 = 1\)
   
   (d) In general, the equation \(ax^2 + bxy + cy^2 + dx + ey + f = 0\) for \(a, b, c, d, e, f \in \mathbb{R}\) defines an ellipse provided \(b^2 - 4ac < 0\). For (b) and (c), \(b^2 - 4ac\) equals \(-40\) and \(-12\) respectively, thus they are ellipses.

2. Do Exercise 3.1.12. in [OS18].

   (a) We have
   \[
   \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2) = \frac{1}{4} (\langle u + v, u + v \rangle - \langle u - v, u - v \rangle)
   \]
   \[
   = \frac{1}{4} (\|u\|^2 + 2 \langle u, v \rangle + \|v\|^2 - \|u\|^2 - 2 \langle u, v \rangle - \|v\|^2)
   \]
   \[
   = \langle u, v \rangle.
   \]
(b) Using the formula above, when \( u = (u_1, u_2)^T \) and \( v = (v_1, v_2)^T \) we have
\[
\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2)
\]
\[
= \frac{1}{4}\left[((u_1 + v_1)^2 - 3(u_1 + v_1)(u_2 + v_2) + 5(u_2 + v_2)^2)
\right.
\]
\[
- ((u_1 - v_1)^2 - 3(u_1 - v_1)(u_2 - v_2) + 5(u_2 - v_2)^2)\]
\[
= \frac{1}{4}(4u_1v_1 - 6u_2v_1 - 6u_1v_2 + 20u_2v_2)
\]
\[
= u_1v_1 - \frac{3}{2}u_2v_1 - \frac{3}{2}u_1v_2 + 5u_2v_2.
\]

3. Do Exercise 3.1.13. in [OS18].

(a) We have
\[
\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y \rangle + \langle x - y, x - y \rangle
\]
\[
= \|x\|^2 + 2 \langle x, y \rangle + \|y\|^2 + \|x\|^2 - 2 \langle x, y \rangle + \|y\|^2
\]
\[
= 2(\|x\|^2 + \|y\|^2).
\]

(b) This is the Parallelogram Law, e.g. see [https://en.wikipedia.org/wiki/Parallelogram_law](https://en.wikipedia.org/wiki/Parallelogram_law).

4. Do Exercise 3.1.22. in [OS18].

(a) \( \langle f, g \rangle = \frac{3}{4}, \|f\| = \frac{1}{\sqrt{3}}, \|g\| = \frac{2\sqrt{7}}{\sqrt{15}} \)

(b) \( \langle f, g \rangle = 0, \|f\| = \frac{2\sqrt{7}}{\sqrt{3}}, \|g\| = \frac{2\sqrt{11}}{\sqrt{15}} \)

(c) \( \langle f, g \rangle = \frac{8}{15}, \|f\| = \frac{1}{2}, \|g\| = \frac{\sqrt{7}}{\sqrt{6}} \)

5. Do Exercise 3.2.26. in [OS18].

(a) It is because \( \int_0^1 p_1p_2 dx = \int_0^1 p_1p_3 dx = \int_0^1 p_2p_3 dx = 0 \) by direct calculation.

(b) We have \( \sin m\pi x \sin n\pi x = \frac{1}{2}(\cos(m - n)\pi x - \cos(m + n)\pi x) \). Thus for \( m, n \in \mathbb{Z}_{>0} \) such that \( m \neq n \), we have
\[
\int_0^1 \sin m\pi x \sin n\pi x dx = \left. \frac{\sin(m - n)\pi x}{2\pi(m - n)} - \frac{\sin(m + n)\pi x}{2\pi(m + n)} \right|_0^1 = 0,
\]
from which the result follows.
6. Do Exercise 3.2.40. in [OS18]. If true, give a proof. If false, provide a counterexample.

By triangle inequality \(\|x + y\| \leq \|x\| + \|y\|\) for \(x = -v\) and \(y = v + w\), we have

\[
\|w\| \leq \|-v\| + \|v + w\| = \|v\| + \|v + w\|
\]

as desired.

References