1. Do Exercise 4.1.5. in [OS18].

Note that two given vectors are linearly independent as long as $a \neq 0$.

(a) $\langle \left( \frac{a}{1} \right), \left( -\frac{a}{1} \right) \rangle = 1 - a^2 = 0 \therefore a = \pm 1$

(b) $\langle \left( \frac{a}{1} \right), \left( -\frac{a}{1} \right) \rangle = 2 - 3a^2 = 0 \therefore a = \pm \sqrt{2/3}$

(c) $\langle \left( \frac{a}{1} \right), \left( -\frac{a}{1} \right) \rangle = 3 - 2a^2 = 0 \therefore a = \pm \sqrt{3/2}$

2. Do Exercise 4.1.13. in [OS18].

(a) It is clear as $A^T K A$ is the Gram matrix associated with the column vectors of $A$ with respect to the inner product defined by $K$.

(b) Let $A$ be the matrix whose columns are the elements in the given basis. Then $K = (A^T)^{-1} A^{-1}$ is the Gram matrix associated with the column vectors of $A^{-1}$, thus it is positive definite. Now we have $A^T K A = I$, thus by part (a) the columns comprise an orthonormal basis with respect to the inner product defined by $K$. This inner product is uniquely determined since $A^T K A = I$ if and only if $K = (A^T)^{-1} A^{-1}$, which implies that $K$ is uniquely determined by $A$.

(c) As $K = (A^T)^{-1} A^{-1}$ by part (b), we have $K = \left( \begin{array}{ccc} 3 & -1 & 3 \\ -2 & 1 & -2 \\ 0 & 1 & 1 \end{array} \right) = \left( \begin{array}{ccc} 10 & -7 & 10 \\ -7 & 5 & -7 \\ 1 & 1 & 0 \end{array} \right)$.

(d) Similarly, we have $K = \left( \begin{array}{ccc} 1 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{array} \right) = \left( \begin{array}{ccc} 3 & -2 & 0 \\ -2 & 6 & -3 \\ 0 & -3 & 2 \end{array} \right)$.

3. Do Exercise 4.1.22. in [OS18].
(a) Let
\[
A = \begin{pmatrix}
\frac{3}{5} & -\frac{4}{13} & -\frac{48}{65} \\
0 & 12/13 & -\frac{5}{13} \\
4/5 & 3/13 & 36/65
\end{pmatrix}.
\]
Then direct calculation shows that $A^T A = I$, thus the column vectors of $A$ form an orthonormal basis.

(b) The coordinates of $v$ is given by $(\langle v, v_1 \rangle, \langle v, v_2 \rangle, \langle v, v_3 \rangle)$, which is equivalent to $A^T v = (7/5, 11/13, -37/65)^T$.

(c) $\|v\| = \sqrt{3} = \sqrt{(7/5)^2 + (11/13)^2 + (-37/65)^2}$.

4. Do Exercise 4.2.3. in [OS18].

One obtains a zero vector on the third step, as the third vector is in the span of the first two vectors.

5. Do Exercise 4.2.6. (b)–(e) in [OS18].

(b) Let $A$ be the given matrix. Then its row canonical form is
\[
\begin{pmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & -2 & 1
\end{pmatrix},
\]
thus a basis of $\ker A$ may be chosen to be $\{(1, 0, 1, 0)^T, (1, -1, 0, 1)^T\}$. Now the Gram-Schmidt process yields $\{\frac{1}{\sqrt{6}}(1, 2, 1, 0)^T, \frac{1}{\sqrt{6}}(1, 0, 1, 2)^T\}$.

(c) From the row canonical form of $A$ above, we may choose a basis of $\text{coim} A$ to be $\{(1, 0, 1, -1)^T, (0, 1, -2, 1)^T\}$. Now the Gram-Schmidt process yields $\{\frac{1}{\sqrt{3}}(1, 0, 1, -1)^T, \frac{1}{\sqrt{3}}(1, 1, -1, 0)^T\}$.

(d) Let $B$ be the given matrix. Then the row canonical form of $B^T$ is
\[
\begin{pmatrix}
1 & 0 & -\frac{2}{3} & 4 \\
0 & 1 & \frac{1}{3} & -3 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
thus we may choose a basis of $\text{im} B$ to be $\{(1, 0, -2/3, 4^T), (0, 1, 1/3, -3)^T\}$. Now the Gram-Schmidt process yields $\{\frac{1}{\sqrt{157}}(3, 0, -2, 12)^T, \frac{1}{9\sqrt{471}}(110, 157, -21, -31)^T\}$.

(e) From the row canonical form of $B^T$ above, we may choose a basis of $\text{coker} B$ to be $\{(2/3, -1/3, 1, 0)^T, (-4, 3, 0, 1)^T\}$. Now the Gram-Schmidt process yields $\{\frac{1}{\sqrt{14}}(2, -1, 3, 0)^T, \frac{1}{\sqrt{942}}(-34, 31, 33, 14)^T\}$.

References