1. Suppose that we are given an \( n \) by \( m \) matrix \( A \) whose column vectors are linearly independent an \( n \) by \( n \) symmetric positive definite matrix \( C \). Set \( P = A(A^TCA)^{-1}A^T \).

(a) Show that \( A^TCA \) is invertible. (Thus, \( P \) is well-defined.)

It is invertible as \( A^TCA \) is the Gram matrix of the column vectors of \( A \) with respect to the inner product \( \langle x, y \rangle = x^T Cy \) and \( A \) has linearly independent columns by assumption.

(b) Prove that \( P^2 = P \) and \( C^{-1}P^T C = P \).

We have
\[
P^2 = A(A^TCA)^{-1}A^TCA(A^TCA)^{-1}A^T C = A(A^TCA)^{-1}A^T C = P.
\]
Also,
\[
C^{-1}P^T C = C^{-1}(CA(A^TCA)^{-1}A^T)C = A(A^TCA)^{-1}A^T C = P.
\]
Here we use the fact that \( C \) is symmetric.

(c) Prove that \( \text{im} \ P = \text{im} \ A \).

As \( P v = A((A^TCA)^{-1}A^T C)v \), we have \( \text{im} \ P \subset \text{im} \ A \). On the other hand, for any \( Av \in \text{im} \ A \), we have \( PA v = A(A^TCA)^{-1}A^T C Av = Av \), thus \( Av = PAv \in \text{im} \ P \), i.e. \( \text{im} \ A \subset \text{im} \ P \). Thus we have \( \text{im} \ P = \text{im} \ A \).

(d) Prove that \( P = I \) if \( A \) is a square matrix.

If \( A \) is square then \( A \) is invertible, thus
\[
P = A(A^TCA)^{-1}A^T C = A(A^{-1}C^{-1}(A^T)^{-1})A^T C = I.
\]
Or, as \( A \) is invertible we have \( \text{im} \ P = \text{im} \ A = \mathbb{R}^n \) which means that \( P \) is also invertible. As \( P^2 = P \), we should have \( P = I \).

(e) Prove that \( (v - Pv) \perp \text{im} \ A \) for any \( v \in \mathbb{R}^n \) with respect to the inner product \( \langle x, y \rangle = x^T Cy \).
We need to show that \( \langle v - Pv, Aw \rangle = 0 \) for any \( v \in \mathbb{R}^n \) and \( w \in \mathbb{R}^m \). It is equivalent to that
\[
(v - Pv)^T CAw = 0 \quad \forall v \in \mathbb{R}^n, w \in \mathbb{R}^m
\]
\[
\Leftrightarrow v^T(I - P)^T CAw = 0 \quad \forall v \in \mathbb{R}^n, w \in \mathbb{R}^m
\]
\[
\Leftrightarrow (I - P)^T CA = 0 \Leftrightarrow (I - P^T)CA = 0
\]
\[
\Leftrightarrow CA = P^T CA \Leftrightarrow A = C^{-1}P^T CA \Leftrightarrow A = PA
\]
where the last step follows from (b). But \( PA = A(A^T CA)^{-1}A^T CA = A \), thus the statements above are all true.

2. Do Exercise 4.4.11. in OS18.

(a) \( P = \begin{pmatrix}
1 & -1 & -7 \\
-4 & 4 & 20 \\
-4 & 20 & 100 \\
-7 & 20 & 100
\end{pmatrix},
\)
\( Pv = \begin{pmatrix}
1 \\
-3 \\
-7 \\
10
\end{pmatrix} \)

(b) \( P = \begin{pmatrix}
1 & -1 & -7 \\
-4 & 4 & 20 \\
-4 & 20 & 100 \\
-7 & 20 & 100
\end{pmatrix},
\)
\( Pv = \begin{pmatrix}
1 \\
-3 \\
-7 \\
10
\end{pmatrix} \)

(c) \( P = \begin{pmatrix}
1 & -1 & -7 \\
-4 & 4 & 20 \\
-4 & 20 & 100 \\
-7 & 20 & 100
\end{pmatrix},
\)
\( Pv = \begin{pmatrix}
1 \\
-3 \\
-7 \\
10
\end{pmatrix} \)

(d) \( P = \begin{pmatrix}
1 & -1 & -7 \\
-4 & 4 & 20 \\
-4 & 20 & 100 \\
-7 & 20 & 100
\end{pmatrix},
\)
\( Pv = \begin{pmatrix}
1 \\
-3 \\
-7 \\
10
\end{pmatrix} \)

3. Do Exercise 5.2.1. in OS18.

If we set \( v = (x, y, z)^T \), then \( f(x, y, z) = v^T \begin{pmatrix}
1 & 1 & 0 \\
1 & 3 & 1 \\
0 & 1 & 1
\end{pmatrix} v - 2v^T \begin{pmatrix}
1 \\
0 \\
-3/2
\end{pmatrix} + 2 \). Thus
\[
\text{its global minimum is } 2 - \begin{pmatrix}
1 & 0 \\
0 & 3/2
\end{pmatrix} \begin{pmatrix}
1 & 1 & 0 \\
1 & 3 & 1 \\
0 & 1 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
1 \\
0 \\
-3/2
\end{pmatrix} = -3/2. \text{ This is the global minimum by Theorem 5.2.}
4. Do Exercise 5.2.9. in [OS18].

By Theorem 5.2, its global minimum is \(-f^* K^{-1} f\) which is always nonpositive since \(K\) is positive definite. By the same reason, it is zero if and only if \(f = 0\), i.e. \(p(x)\) is homogeneous of degree 2.

5. Do Exercise 5.3.1. in [OS18].

If we set \(A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 3 \end{pmatrix}\) and \(v = (1, 1, 1)^T\), then the closest point is given by 
\[
A(A^T A)^{-1} A^T v = \left(\frac{6}{7}, \frac{38}{35}, \frac{36}{35}\right)^T.
\]
Also its distance from \(v\) is given by 
\[
\sqrt{\left(-\frac{1}{7}\right)^2 + \left(\frac{3}{35}\right)^2 + \left(\frac{1}{35}\right)^2} = \frac{1}{\sqrt{35}}.
\]

6. Do Exercise 5.3.2. in [OS18].

(a) We set \(C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}\). Then the closest point is given by 
\[
A(A^T CA)^{-1} A^T Cv = \left(\frac{151}{181}, \frac{190}{181}, \frac{185}{181}\right)^T
\]
and its distance from \(v\) is equal to 
\[
\sqrt{2 \left(-\frac{30}{181}\right)^2 + 4 \left(\frac{9}{35}\right)^2 + 3 \left(\frac{4}{35}\right)^2} = \frac{\sqrt{12}}{181}.
\]

(b) We set \(C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\). Then the closest point is 
\[
A(A^T CA)^{-1} A^T Cv = \left(\frac{6}{7}, \frac{15}{14}, \frac{15}{14}\right)^T
\]
and its distance from \(v\) is equal to 
\[
\sqrt{(v - Pv)^T C(v - Pv)} = \frac{1}{\sqrt{14}}.
\]

— More Exercises Suggestions (these are not a part of homework): 4.4.9, 4.4.10, 4.4.15, 4.4.22, 4.4.28, 4.4.29, 4.4.31, 5.2.3, 5.2.4, 5.2.8, 5.2.11, 5.3.4, 5.3.9, 5.3.14, 5.3.15

REFERENCES