PROBLEM SET 10 (DUE: APR 24, WED)

1. Suppose that we are given an n by m matrix A whose column vectors are linearly independent an n by n symmetric positive definite matrix C. Set $P = A(A^TCA)^{-1}A^TC$.

   (a) Show that A^TCA is invertible. (Thus, P is well-defined.)

   > It is invertible as A^TCA is the Gram matrix of the column vectors of A with respect to the inner product $\langle x, y\rangle = x^TCy$ and A has linearly independent columns by assumption.

   (b) Prove that $P^2 = P$ and $C^{-1}P^TC = P$.

   > We have
   > $$P^2 = A(A^TCA)^{-1}A^TCA(A^TCA)^{-1}A^TC = A(A^TCA)^{-1}A^TC = P.$$
   > Also,
   > $$C^{-1}P^TC = C^{-1}(CA(A^TCA)^{-1}A^T)C = A(A^TCA)^{-1}A^TC = P.$$
   > Here we use the fact that C is symmetric.

   (c) Prove that $\operatorname{im} P = \operatorname{im} A$.

   > As $Pv = A((A^TCA)^{-1}A^TCv)$, we have $\operatorname{im} P \subset \operatorname{im} A$. On the other hand, for any $Av \in \operatorname{im} A$, we have $PAv = A(A^TCA)^{-1}A^TCAv = Av$, thus $Av = PAv \in \operatorname{im} P$, i.e. $\operatorname{im} A \subset \operatorname{im} P$. Thus we have $\operatorname{im} P = \operatorname{im} A$.

   (d) Prove that $P = I$ if A is a square matrix.

   > If A is square then A is invertible, thus
   > $$P = A(A^TCA)^{-1}A^TC = A(A^{-1}C^{-1}(A^T)^{-1})A^TC = I.$$
   > Or, as A is invertible we have $\operatorname{im} P = \operatorname{im} A = \mathbb{R}^n$ which means that P is also invertible. As $P^2 = P$, we should have $P = I$.

   (e) Prove that $(v - Pv) \perp \operatorname{im} A$ for any $v \in \mathbb{R}^n$ with respect to the inner product $\langle x, y\rangle = x^TCy$.

We need to show that $\langle v - Pv, Aw\rangle = 0$ for any $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$. It is equivalent to that

$$\begin{aligned} &(v - Pv)^T CAw = 0 \qquad \forall v \in \mathbb{R}^n, w \in \mathbb{R}^m \\ \Leftrightarrow\ & v^T (I - P)^T CAw = 0 \qquad \forall v \in \mathbb{R}^n, w \in \mathbb{R}^m \\ \Leftrightarrow\ & (I - P)^T CA = 0 \Leftrightarrow (I - P^T)CA = 0 \\ \Leftrightarrow\ & CA = P^T CA \Leftrightarrow A = C^{-1}P^T CA \Leftrightarrow A = PA \end{aligned}$$

where the last step follows from (b). But $PA = A(A^T CA)^{-1}A^T CA = A$, thus the statements above are all true.

2. Do Exercise 4.4.11. in [OS18].

(a) $P = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{7}{20} & \frac{1}{20} \\ -\frac{1}{4} & \frac{1}{4} & \frac{7}{20} & -\frac{1}{20} \\ -\frac{7}{20} & \frac{7}{20} & \frac{49}{100} & -\frac{7}{100} \\ \frac{1}{20} & -\frac{1}{20} & -\frac{7}{100} & \frac{1}{100} \end{pmatrix}, Pv = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{7}{20} \\ \frac{1}{20} \end{pmatrix}$

(b) $P = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{7}{9} & -\frac{2}{9} & \frac{1}{9} \\ 0 & -\frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{1}{3} & \frac{1}{9} & -\frac{2}{9} & \frac{7}{9} \end{pmatrix}, Pv = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}$

(c) $P = \begin{pmatrix} \frac{7}{25} & -\frac{2}{5} & \frac{1}{5} & \frac{1}{25} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ \frac{1}{25} & -\frac{1}{5} & -\frac{2}{5} & \frac{18}{25} \end{pmatrix}, Pv = \begin{pmatrix} \frac{7}{25} \\ -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{25} \end{pmatrix}$

(d) $P = \begin{pmatrix} \frac{7}{15} & -\frac{2}{5} & \frac{4}{15} & \frac{2}{15} \\ -\frac{2}{5} & \frac{7}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{4}{15} & \frac{1}{5} & \frac{13}{15} & -\frac{1}{15} \\ \frac{2}{15} & \frac{1}{10} & -\frac{1}{15} & \frac{29}{30} \end{pmatrix}, Pv = \begin{pmatrix} \frac{7}{15} \\ -\frac{2}{5} \\ \frac{4}{15} \\ \frac{2}{15} \end{pmatrix}$

3. Do Exercise 5.2.1. in [OS18].

If we set $v = (x, y, z)^T$, then $f(x, y, z) = v^T \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} v - 2v^T \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix} + 2$. Thus its global minimum is $2 - \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix} = -\frac{3}{2}$. This is the global minimum by Theorem 5.2.

4. Do Exercise 5.2.9. in [OS18].

By Theorem 5.2, its global minimum is $-f^T K^{-1} f$ which is always nonpositive since K is positive definite. By the same reason, it is zero if and only if $f = 0$, i.e. $p(x)$ is homogeneous of degree 2.

5. Do Exercise 5.3.1. in [OS18].

If we set $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 3 \end{pmatrix}$ and $v = (1,1,1)^T$, then the closest point is given by $A(A^TA)^{-1}A^Tv = (\frac{6}{7}, \frac{38}{35}, \frac{36}{35})^T$. Also its distance from v is given by

$$\sqrt{\left(-\frac{1}{7}\right)^2 + \left(\frac{3}{35}\right)^2 + \left(\frac{1}{35}\right)^2} = \frac{1}{\sqrt{35}}.$$

6. Do Exercise 5.3.2. in [OS18].

(a) We set $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Then the closest point is given by $A(A^TCA)^{-1}A^TCv = (\frac{151}{181}, \frac{190}{181}, \frac{185}{181})^T$ and its distance from v is equal to

$$\sqrt{2\left(-\frac{30}{181}\right)^2 + 4\left(\frac{9}{35}\right)^2 + 3\left(\frac{4}{35}\right)^2} = \sqrt{\frac{12}{181}}.$$

(b) We set $C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Then the closest point is $A(A^TCA)^{-1}A^TCv = (\frac{6}{7}, \frac{15}{14}, \frac{15}{14})^T$ and its distance from v is equal to

$$\sqrt{(v - Pv)^T C (v - Pv)} = \frac{1}{\sqrt{14}}.$$

— More Exercises Suggestions (these are not a part of homework): 4.4.9, 4.4.10, 4.4.15, 4.4.22, 4.4.28, 4.4.29, 4.4.31, 5.2.3, 5.2.4, 5.2.8, 5.2.11, 5.3.4, 5.3.9, 5.3.14, 5.3.15

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.