PROBLEM SET 11 (DUE: MAY 1, WED)

1. Do Exercise 5.4.2. in [OS18].

Here we use the formula $v^* = (A^T A)^{-1} A^T b$.

(a) $$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \therefore v^* = \begin{pmatrix} \frac{1}{15} \\ \frac{41}{45} \end{pmatrix}.$$

(b) $$\begin{pmatrix} 4 & -2 \\ 2 & 3 \\ 1 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -1 \\ 2 \end{pmatrix} \therefore v^* = \begin{pmatrix} -\frac{1}{25} \\ -\frac{8}{21} \end{pmatrix}$$

(c) $$\begin{pmatrix} 2 & 1 & -2 \\ 3 & 0 & -2 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \therefore v^* = \begin{pmatrix} \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{pmatrix}$$

(d) $$\begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 0 & 1 & -3 \\ -5 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 3 \end{pmatrix} \therefore v^* = \begin{pmatrix} \frac{1}{3} \\ 2 \\ \frac{3}{4} \end{pmatrix}$$

(e) $$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \therefore v^* = \begin{pmatrix} \frac{1}{3} \\ 2 \\ -\frac{1}{3} \\ -\frac{4}{3} \end{pmatrix}$$

2. Do Exercise 5.5.1. in [OS18].

We use the same technique as above.

(a) $$\begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix} \therefore \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -3 \\ -3 \end{pmatrix} \therefore \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} \frac{19}{10} \\ -\frac{11}{10} \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ -2 \\ 0 \\ 3 \end{pmatrix} \therefore \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} -\frac{7}{5} \\ \frac{19}{10} \end{pmatrix}$

3. Do Exercise 5.5.15. in [OS18].

We use the same technique as above.

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix} \therefore \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix} \therefore \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ Here the error is zero because for any $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ such that x_1, x_2, x_3 are pairwise distinct there exists (a unique) parabola of the form $y = a + bx + cx^2$ which passes through all of such points.

4. Do Exercise 8.2.6. in [OS18].

Since $\phi_A(t) = t^3 + (a^2 + b^2 + c^2)t$, eigenvalues are $0, \pm i\sqrt{a^2 + b^2 + c^2}$. Therefore,

(a) If $a = b = c = 0$, then $A = 0$ thus the only eigenvalue is 0 and $V_0 = \mathbb{C}^3$.

(b) If $(a, b, c) \neq (0, 0, 0)$ but $a^2 + b^2 + c^2 = 0$, then the only eigenvalue is 0 and $V_0 = \operatorname{span}\{(a, b, c)^T\}$.

(c) Otherwise, i.e. $a^2 + b^2 + c^2 \neq 0$, then direct calculation shows that:

$$V_0 = \operatorname{span}\{(a, b, c)^T\}$$

$$V_{i\sqrt{a^2+b^2+c^2}} = \operatorname{span}\{(ab + ci\sqrt{a^2 + b^2 + c^2}, -a^2 - c^2, bc - ai\sqrt{a^2 + b^2 + c^2})^T\}$$

$$V_{-i\sqrt{a^2+b^2+c^2}} = \operatorname{span}\{(ab - ci\sqrt{a^2 + b^2 + c^2}, -a^2 - c^2, bc + ai\sqrt{a^2 + b^2 + c^2})^T\}$$

5. Do Exercise 8.2.7. in [OS18].

> (a) eigenvalues are i and $i-1$, $V_i = \text{span}\{(1,0)^T\}, V_{i-1} = \text{span}\{(-1,1)^T\}$.
>
> (b) eigenvalues are $\pm\sqrt{5}$, $V_{\sqrt{5}} = \text{span}\{(2+\sqrt{5}, -i)^T\}, V_{-\sqrt{5}} = \text{span}\{(2-\sqrt{5}, -i)^T\}$.
>
> (c) eigenvalues are -2 and i, $V_{-2} = \text{span}\{(-1,-2,1)^T\}$,
> $V_i = \text{span}\{(-1+i, 0, 1)^T, (1+i, 1, 0)^T\}$.

6. Do Exercise 8.2.9. in [OS18].

> Let A be an $n \times n$ matrix with every entry equal to 1. As $\text{rank} A = 1$, $\dim \ker A = n-1$ and direct calculation shows that $V_0 = \ker A \supset \{e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n\}$. As $\{e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n\}$ is linearly independent of size $n-1 = \dim V_0$, we see that $V_0 = \text{span}\{e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n\}$. On the other hand, we have $A(e_1 + e_2 + \cdots + e_n) = n(e_1 + e_2 + \cdots + e_n)$ by direct calculation. As $\dim V_n + \dim V_0 \leq n$ and $V_n \ni e_1 + e_2 + \cdots + e_n$, we conclude that $\dim V_n = 1$ and $V_n = \text{span}\{e_1 + e_2 + \cdots + e_n\}$.

— More Exercises Suggestions (these are not a part of homework): 5.4.1, 5.4.6, 5.4.11, 5.4.12, 5.4.13, 5.5.9, 5.5.12, 5.5.16, 5.5.19, 5.5.41, 8.2.1, 8.2.2, 8.2.3, 8.2.5, 8.2.12, 8.2.10, 8.2.13

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.