PROBLEM SET 1 (DUE: FEB 6, WED)

1. Do Exercise 1.2.21. in [OS18].

If $A = 0$, then it is clear that $Av = 0$ for any vector v. For the converse, consider the vector e_i where its entries are 0 except the i-th entry which is 1. Then we have $Ae_i = 0$. However, Ae_i is the same as the i-th column of A by the definition of matrix multiplication, we see that the i-th column of A is zero. Since i is arbitrary, we conclude that $A = 0$.

2. Do Exercise 1.2.23. in [OS18]. (You do not need to justify your answer.)

e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

3. Do Exercise 1.3.21. in [OS18]. (You do not need to write down a rigorous proof, but explain how you get such an LU factorization in each case.)

Here we describe row operations to calculate U and the corresponding L.

(a) $\begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix} \xrightarrow{[2]\leftarrow[2]+[1]} \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \xrightarrow{[2]\leftarrow[2]-3[1]} \begin{pmatrix} 1 & 3 \\ 0 & -8 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow[{[3]\leftarrow[3]-[1]}]{[2]\leftarrow[2]+[1]} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 0 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{[2]\leftarrow[2]-\frac{1}{2}[1]} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & -\frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{[3]\leftarrow[3]-\frac{1}{3}[2]} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{6} \end{pmatrix}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$$

(e) $$\begin{pmatrix}-1 & 0 & 0\\ 2 & -3 & 0\\ 1 & 3 & 2\end{pmatrix} \xrightarrow[\text{[3]}\leftarrow\text{[3]}+\text{[1]}]{\text{[2]}\leftarrow\text{[2]}+2\text{[1]}} \begin{pmatrix}-1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 3 & 2\end{pmatrix} \xrightarrow{\text{[3]}\leftarrow\text{[3]}+\text{[2]}} \begin{pmatrix}-1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$L = \begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & -1 & 1\end{pmatrix}$$

(f) $$\begin{pmatrix}1 & 0 & -1\\ 2 & 3 & 2\\ -3 & 1 & 0\end{pmatrix} \xrightarrow[\text{[3]}\leftarrow\text{[3]}+3\text{[1]}]{\text{[2]}\leftarrow\text{[2]}-2\text{[1]}} \begin{pmatrix}1 & 0 & -1\\ 0 & 3 & 4\\ 0 & 1 & -3\end{pmatrix} \xrightarrow{\text{[3]}\leftarrow\text{[3]}-\frac{1}{3}\text{[2]}} \begin{pmatrix}1 & 0 & -1\\ 0 & 3 & 4\\ 0 & 0 & -\frac{13}{3}\end{pmatrix}$$

$$L = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -3 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & \frac{1}{3} & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -3 & \frac{1}{3} & 1\end{pmatrix}$$

(g) $$\begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 2 & -1 & -1\\ -1 & 3 & 0 & 2\\ 0 & -1 & 2 & 1\end{pmatrix} \xrightarrow{\text{[3]}\leftarrow\text{[3]}+\text{[1]}} \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 2 & -1 & -1\\ 0 & 3 & -1 & 2\\ 0 & -1 & 2 & 1\end{pmatrix}$$

$$\xrightarrow[\text{[4]}\leftarrow\text{[4]}+\frac{1}{2}\text{[2]}]{\text{[3]}\leftarrow\text{[3]}-\frac{3}{2}\text{[2]}} \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 2 & -1 & -1\\ 0 & 0 & \frac{1}{2} & \frac{7}{2}\\ 0 & 0 & \frac{3}{2} & \frac{1}{2}\end{pmatrix} \xrightarrow{\text{[4]}\leftarrow\text{[4]}-3\text{[3]}} \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 2 & -1 & -1\\ 0 & 0 & \frac{1}{2} & \frac{7}{2}\\ 0 & 0 & 0 & -10\end{pmatrix}$$

$$L = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & \frac{3}{2} & 1 & 0\\ 0 & -\frac{1}{2} & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 3 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1 & \frac{3}{2} & 1 & 0\\ 0 & -\frac{1}{2} & 3 & 1\end{pmatrix}$$

(h) $$\begin{pmatrix}1 & 1 & -2 & 3\\ -1 & 2 & 3 & 0\\ -2 & 1 & 1 & -2\\ 3 & 0 & 1 & 5\end{pmatrix} \xrightarrow{\substack{\text{[2]}\leftarrow\text{[2]}+\text{[1]}\\ \text{[3]}\leftarrow\text{[3]}+2\text{[1]}\\ \text{[4]}\leftarrow\text{[4]}-3\text{[1]}}} \begin{pmatrix}1 & 1 & -2 & 3\\ 0 & 3 & 1 & 3\\ 0 & 3 & -3 & 4\\ 0 & -3 & 7 & -4\end{pmatrix}$$

$$\xrightarrow[\text{[4]}\leftarrow\text{[4]}+\text{[2]}]{\text{[3]}\leftarrow\text{[3]}-\text{[2]}} \begin{pmatrix}1 & 1 & -2 & 3\\ 0 & 3 & 1 & 3\\ 0 & 0 & -4 & 1\\ 0 & 0 & 8 & -1\end{pmatrix} \xrightarrow{\text{[4]}\leftarrow\text{[4]}+2\text{[3]}} \begin{pmatrix}1 & 1 & -2 & 3\\ 0 & 3 & 1 & 3\\ 0 & 0 & -4 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$L = \begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -2 & 0 & 1 & 0\\ 3 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & -1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -2 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -2 & 1 & 1 & 0\\ 3 & -1 & -2 & 1\end{pmatrix}$$

(i) $$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 4 & 0 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} \xrightarrow[{[4]\leftarrow[4]-\frac{1}{2}[1]}]{\substack{[2]\leftarrow[2]-\frac{1}{2}[1] \\ [3]\leftarrow[3]-\frac{3}{2}[1] \\ [4]\leftarrow[4]-\frac{1}{2}[1]}} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & \frac{7}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & -\frac{5}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$\xrightarrow{\substack{[3]\leftarrow[3]+\frac{3}{7}[2] \\ [4]\leftarrow[4]-\frac{1}{7}[2]}} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & \frac{7}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{22}{7} & \frac{5}{7} \\ 0 & 0 & \frac{5}{7} & \frac{10}{7} \end{pmatrix} \xrightarrow{[4]\leftarrow[4]+\frac{5}{22}[3]} \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & \frac{7}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{22}{7} & \frac{5}{7} \\ 0 & 0 & 0 & \frac{35}{22} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{7} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{5}{22} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{7} & 1 & 0 \\ \frac{1}{2} & \frac{1}{7} & -\frac{5}{22} & 1 \end{pmatrix}$$

4. Do Exercise 1.3.27. in [OS18]. If it is true, then provide a proof. Otherwise, give a counterexample.

The answer is false. For example, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has a zero on the diagonal, but $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ is regular.

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.