PROBLEM SET 2 (DUE: FEB 13, WED)

1. Do Exercise 1.4.10. in [OS18].

(a) $P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(b) $P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(c) No, since $P_1P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = P_2P_1$.

(d) P_1P_2 moves the first row to the second row, the second row to the fourth row, and the fourth row to the first row. P_2P_1 moves the first row to the fourth row, the fourth row to the second row, and the second row to the first row.

2. Do Exercise 1.4.11. in [OS18].

(a) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(c) $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(d) $P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

3. Do Exercise 1.4.21. in [OS18]. (You do not need to write down a rigorous proof, but please show each step of your calculation.)

(a) $\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \xrightarrow{[1]\leftrightarrow[2]} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, L = I_2$.

$$Ux = Pb = \left(2, 3\right)^T \Rightarrow x = \left(\tfrac{5}{2}, 3\right)^T.$$

(b) $\begin{pmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 1 & 7 \end{pmatrix} \xrightarrow{[1]\leftrightarrow[2]} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & 1 & 7 \end{pmatrix} \xrightarrow{[2]\leftrightarrow[3]} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & -4 \end{pmatrix}$, and

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, L = I_3.$$

$$Ux = Pb = \left(2, -1, 1\right)^T \Rightarrow x = \left(\tfrac{5}{4}, \tfrac{3}{4}, -\tfrac{1}{4}\right)^T.$$

(c) $\begin{pmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{[1]\leftrightarrow[3]} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{[3]\leftarrow[3]-\frac{1}{2}[2]} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{9}{2} \end{pmatrix}$, and

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

$$LUx = Pb = \left(-1, 2, 1\right)^T \Rightarrow Ux = \left(-1, 2, 0\right)^T \Rightarrow x = \left(-1, 1, 0\right)^T.$$

(d) $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 6 & 2 & -1 \\ 1 & 1 & -7 & 2 \\ 1 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{\substack{[2]\leftarrow[2]-3[1] \\ [3]\leftarrow[3]-[1] \\ [4]\leftarrow[4]-[1]}} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 5 & -1 \\ 0 & -1 & -6 & 2 \\ 0 & -3 & 3 & 1 \end{pmatrix} \xrightarrow{[2]\leftrightarrow[3]} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & -6 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & -3 & 3 & 1 \end{pmatrix}$

$\xrightarrow{[4]\leftarrow[4]-3[2]} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & -6 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 21 & -5 \end{pmatrix} \xrightarrow{[4]\leftarrow[4]-\frac{21}{5}[3]} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & -6 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & -\frac{4}{5} \end{pmatrix}$, and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L = P\begin{pmatrix}1&0&0&0\\3&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&3&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{21}{5}&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\1&1&0&0\\3&0&1&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&3&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{21}{5}&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\3&0&1&0\\1&3&\frac{21}{5}&1\end{pmatrix}.$$

$$LUx = Pb = \big(1,0,0,3\big)^T \Rightarrow Ux = \big(1,-1,-3,\tfrac{88}{5}\big)^T \Rightarrow x = \big(22,-13,-5,-22\big)^T$$

(e) $$\begin{pmatrix}0&1&0&0\\2&3&1&0\\1&4&-1&2\\7&-1&2&3\end{pmatrix} \xrightarrow{[1]\leftrightarrow[3]} \begin{pmatrix}1&4&-1&2\\2&3&1&0\\0&1&0&0\\7&-1&2&3\end{pmatrix} \xrightarrow[{[4]\leftarrow[4]-7[1]}]{[2]\leftarrow[2]-2[1]} \begin{pmatrix}1&4&-1&2\\0&-5&3&-4\\0&1&0&0\\0&-29&9&-11\end{pmatrix}$$

$$\xrightarrow{[2]\leftrightarrow[3]} \begin{pmatrix}1&4&-1&2\\0&1&0&0\\0&-5&3&-4\\0&-29&9&-11\end{pmatrix} \xrightarrow[{[4]\leftarrow[4]+29[2]}]{[3]\leftarrow[3]+5[2]} \begin{pmatrix}1&4&-1&2\\0&1&0&0\\0&0&3&-4\\0&0&9&-11\end{pmatrix}$$

$$\xrightarrow{[4]\leftarrow[4]-3[3]} \begin{pmatrix}1&4&-1&2\\0&1&0&0\\0&0&3&-4\\0&0&0&1\end{pmatrix}, \text{ and}$$

$$P = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix},$$

$$L = P\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\2&1&0&0\\0&0&1&0\\7&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-5&1&0\\0&-29&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\2&1&0&0\\0&0&1&0\\7&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-5&1&0\\0&-29&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\0&1&0&0\\2&0&1&0\\7&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-5&1&0\\0&-29&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\2&-5&1&0\\7&-29&3&1\end{pmatrix}.$$

$$LUx = Pb = \big(0,-1,-4,5\big)^T \Rightarrow Ux = \big(0,-1,-9,3\big)^T \Rightarrow x = \big(-1,-1,1,3\big)^T$$

(f) $$\begin{pmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 1 & -7 & 2 & 3 \\ 1 & 4 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix} \xrightarrow{[1]\leftrightarrow[3]} \begin{pmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -7 & 2 & 3 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 & 3 \end{pmatrix}$$

$$\xrightarrow[{[5]\leftarrow[5]-\frac{1}{2}[3]}]{[4]\leftarrow[4]-\frac{1}{2}[3]} \begin{pmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -7 & 2 & 3 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{11}{2} & 1 \end{pmatrix} \xrightarrow{[5]\leftarrow[5]+\frac{11}{3}[4]} \begin{pmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -7 & 2 & 3 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and}$$

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L = P \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{11}{3} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{11}{3} & 1 \end{pmatrix}.$$

$$LUx = Pb = \big(0,-2,-3,0,7\big)^T \Rightarrow Ux = \big(0,-2,-3,\tfrac{3}{2},0\big)^T \Rightarrow x = \big(1,0,0,-1,0\big)^T$$

4. Do Exercise 1.5.2. in [OS18], using Gauss-Jordan elimination.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & -1 & -8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{[3]\to[3]-[1]} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -3 & -8 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{[3]\to[3]+3[2]} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right) \xrightarrow{[2]\to[2]-3[3]} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -8 & -3 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{[1]\to[1]-2[2]} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 16 & 6 \\ 0 & 1 & 0 & 3 & -8 & -3 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right)$$

Thus the right inverse of A is $X = \left(\begin{smallmatrix} -5 & 16 & 6 \\ 3 & -8 & -3 \\ -1 & 3 & 1 \end{smallmatrix}\right)$. As direct calculation shows that $XA = I_3$, X is also the left inverse of A.

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.