PROBLEM SET 3 (DUE: FEB 20, WED)

1. Do Exercise 1.5.4. in [OS18].

> L^{-1} can be shown to be the inverse of L by direct calculation. However, we have
> $$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b+ac & -c & 1 \end{pmatrix}.$$
> In fact, L is a special case of M when $c = 0$.

2. Do Exercise 1.5.19. in [OS18].

> (a) $A \sim A$ since $A = I^{-1}AI$.
>
> (b) If $A \sim B$ then there exists S such that $B = S^{-1}AS$. But it means that $A = (S^{-1})^{-1}B(S^{-1})$, thus $B \sim A$.
>
> (c) If $A \sim B$ and $B \sim C$, then there exist S, T such that $B = S^{-1}AS$ and $C = T^{-1}BT$. But it means that $C = (ST)^{-1}A(ST)$, thus $A \sim C$.

3. Do Exercise 1.6.7. in [OS18].

> Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^TA = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$. Similarly, $AA^T = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$. As these are supposed to be the same matrix, we have $b^2 = c^2$ and $ac+bd = ab+cd$, i.e. $[b = c]$ or $[b = -c$ and $a = d]$. In other words, we have $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ for some $a, b, d \in \mathbb{R}$ (i.e. symmetric matrix) or $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$.

4. Do Exercise 1.6.13. in [OS18]. Here you may use 1.6.12. (Hint for (b): there exists such a matrix of size 2 by 2.)

> (a) Let e_j be the vector defined in 1.6.12. Then $e_i^TAe_j$ is equal to the (i, j) entry of A. As $e_i^TAe_j = e_i^TBe_j$, the (i, j) entries of A and B are the same. As i and j are arbitrary, we see that $A = B$.
>
> (b) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then for any $v = \begin{pmatrix} a \\ b \end{pmatrix}$, we have $v^TAv = 0 = v^TBv$, however $A \neq B$.

5. Do Exercise 1.6.26. in [OS18].

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{pmatrix}$, thus $D = \begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$.

(b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$, thus $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix}$, thus $D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{pmatrix}$.

6. Do Exercise 1.8.7. (a)–(f) in [OS18].

(a) $\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}$, thus its rank is 2.

(b) $\begin{pmatrix} 2 & 1 & 3 \\ -2 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, thus its rank is 1.

(c) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, thus its rank is 2.

(d) $\begin{pmatrix} 2 & -1 & 0 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{3}{2} & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$, thus its rank is 3.

(e) $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \to \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, thus its rank is 1.

(f) It is already in row echelon form, thus its rank is 1.

— More Exercises Suggestions (these are not a part of homework): 1.5.11, 1.5.14, 1.5.20, 1.5.21, 1.5.29, 1.6.9, 1.6.12, 1.6.23, 1.6.28, 1.8.4, 1.8.10, 1.8.13, 1.8.17, 1.8.18, 1.8.19, 1.8.21

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.