PROBLEM SET 4 (DUE: FEB 27, WED)

1. Do Exercise 1.8.24. in [OS18].

> U has at least one 0 on the diagonal if and only if $\det U = 0$ if and only if $\ker U \supsetneq \{0\}$.

2. Do Exercise 1.9.6. in [OS18].

> $\det cA = \det cI \det A = c^n \det A$ since $\det cI = c^n$.

3. Do Exercise 1.9.19. in [OS18]. (You do not need to write down the general formula.)

> We have
>
> (a) $t_2 - t_1$
>
> (b) $t_3t_1^2 + t_2^2t_1 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2 - t_2^2t_3$
>
> (c) $-t_2t_3^2t_1^3 + t_2t_4^2t_1^3 - t_3t_4^2t_1^3 + t_2^2t_3t_1^3 - t_2^2t_4t_1^3 + t_3^2t_4t_1^3 + t_2t_3^3t_1^2 - t_2t_4^3t_1^2 + t_3t_4^3t_1^2 - t_2^3t_3t_1^2 + t_2^3t_4t_1^2 - t_3^3t_4t_1^2 - t_2^2t_3^3t_1 + t_2^2t_4^3t_1 - t_3^2t_4^3t_1 + t_2^3t_3^2t_1 - t_2^3t_4^2t_1 + t_3^3t_4^2t_1 + t_2t_3^2t_4^3 - t_2^2t_3t_4^3 - t_2t_3^3t_4^2 + t_2^3t_3t_4^2 + t_2^2t_3^3t_4 - t_2^3t_3^2t_4$

4. Do Exercise 2.2.23. in [OS18].

> We have
>
> (i) $0 \in W_i$ for any W_i, thus $0 \in \bigcap W_i$.
>
> (ii) For any $v, w \in \bigcap W_i$ and for any $a \in \mathbb{R}$, we have $av + w \in W_i$ for any W_i, thus $av + w \in \bigcap W_i$.
>
> Thus $\bigcap W_i$ is a vector subspace of V.

5. Do Exercise 2.3.24. in [OS18].

> (a) They are linearly dependent since no more than 2 vectors can be linearly independent in $\mathbb{R}^2$.
>
> (b) They are linearly dependent since $2(1, 2, -1) - (2, 4, -2) = 0$.
>
> (c) They are linearly independent since the matrix A with row vectors $(1, 2, 3)$, $(1, 4, 8)$, $(1, 5, 7)$ has determinant $-7 \neq 0$.

(d) They are linearly dependent since no more than 3 vectors can be linearly independent in $\mathbb{R}^3$.

(e) They are linearly independent since the row reduced echelon form of the matrix A with row vectors $(1,2,0,3)$, $(-3,-1,2,-2)$, $(3,-4,-4,5)$ is $\begin{pmatrix} 1 & 0 & -\frac{4}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ which means that $\operatorname{rank} A = 3$.

(f) They are linearly independent since the row reduced echelon form of the matrix A with row vectors $(2,1,-1,3)$, $(-1,3,1,0)$, $(5,1,2,-3)$ is $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{11} \\ 0 & 1 & 0 & \frac{8}{11} \\ 0 & 0 & 1 & -\frac{23}{11} \end{pmatrix}$ which means that $\operatorname{rank} A = 3$.

6. Do Exercise 2.4.10. in [OS18].

(a) basis: $\{(3,1,-1)^T\}$, dimension: 1

(b) basis: $\{(2,0,1)^T, (0,-1,3)^T\}$, dimension: 2

(c) basis: $\{(1,0,-1,2)^T, (0,1,1,3)^T, (1,-2,1,1)^T\}$, dimension: 3

7. Do Exercise 2.4.21. in [OS18].

We only need to show that $\{Av_1, Av_2, \ldots, Av_n\}$ is linearly independent. But if $a_1 Av_1 + a_2 Av_2 + \cdots + a_n Av_n = 0$, then by multiplying A^{-1} on the left we have $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$, which implies that $a_1 = a_2 = \cdots = a_n = 0$ since $\{v_1, v_2, \ldots, v_n\}$ is assumed to be linearly independent. Thus $\{Av_1, Av_2, \ldots, Av_n\}$ is linearly independent and the claim follows.

— More Exercises Suggestions (these are not a part of homework): 1.8.22, 1.8.27, 1.9.1, 1.9.3, 1.9.4, 1.9.5, 1.9.16, 1.9.17, 2.1.2, 2.1.10, 2.1.12, 2.2.3, 2.2.7, 2.2.11, 2.2.15, 2.2.29, 2.3.2, 2.3.5, 2.3.8, 2.3.16, 2.3.21, 2.3.28, 2.3.31, 2.3.33, 2.3.39, 2.4.3, 2.4.4, 2.4.15, 2.4.25, 2.4.26, 2.4.27

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.