1. Do Exercise 2.5.10. in [OS18].

\[ Cx = 0 \Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} x = 0 \Rightarrow \begin{pmatrix} Ax \\ Bx \end{pmatrix} = 0 \Rightarrow Ax = 0, \text{ thus } \ker C \subset \ker A. \text{ Also if } A = (0) \text{ and } C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ then } \ker C = \{(0)\} \neq \mathbb{R} = \ker A. \]

2. Do Exercise 2.5.11. in [OS18].

It is clear since \( \text{im} C = \text{span}\{\text{columns of } C\} \supset \text{span}\{\text{columns of } A\} = \text{im} A. \) Also if \( A = (0) \) and \( C = (0 \ 1) \) then \( \text{im} C = \mathbb{R} \neq \{(0)\} = \text{im} A. \)

3. Do Exercise 2.5.14. in [OS18].

(a) follows from direct calculation. For (b), note that the columns are not all parallel to one another, thus \( \text{rank} A \geq 2 \) which means \( \text{dim} \ker A \leq 1 \) by the fundamental theorem of linear algebra. As \( x_1^* - x_2^* = (4, -2, 2)^T \) satisfies \( A(x_1^* - x_2^*) = 0 \), it follows that \( (4, -2, 2)^T \in \ker A. \) Thus we conclude that \( \text{dim} \ker A = 1 \) and \( \ker A = \text{span}\{(4, -2, 2)^T\}. \) It means that the general solution is given by \( \ker A + x_1^* = \{t(4, -2, 2)^T + (1, 1, 0)^T \mid t \in \mathbb{R}\}. \)

4. Do Exercise 2.5.21. in [OS18].

(a) (i) \( \text{span}\{(1, 2)^T\} \) (ii) \( \text{span}\{(1, -3)^T\} \) (iii) \( \text{span}\{(3, 1)^T\} \) (iv) \( \text{span}\{(-2, 1)^T\} \)

(b) (i) \( \text{span}\{(1, 0, -1)^T, (0, 1, 2)^T\} \) (ii) \( \text{span}\{(1, 2, 0)^T, (0, 0, 1)^T\} \)

(c) (i) \( \text{span}\{(1, 0, 3)^T, (0, 1, -1)^T\} \) (ii) \( \text{span}\{(1, 0, -1, 3)^T, (0, 1, 3, -2)^T\} \)

(d) (i) \( \text{span}\{(1, 0, 2, 0, -2)^T, (0, 1, 1, 0, -1)^T, (0, 0, 0, 1, 1)^T\} \)

5. Do Exercise 2.5.30. in [OS18].
(a) It is obvious as $A = A^T$.

(b) $\text{im}\, A = \text{coim}\, A = \text{span}\{(1, 0, -2)^T, (0, 1, 1)^T\}$, $\ker\, A = \text{coker}\, A = \text{span}\{(2, -1, 1)^T\}$.

(c) No. If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $A^T = -A$, thus $\text{coker}\, A = \text{ker}\, A^T = \text{ker}\, (-A) = \text{ker}\, A$ and $\text{coim}\, A = \text{im}\, A^T = \text{im}\, (-A) = \text{im}\, A$. However $A$ is not symmetric.

— More Exercises Suggestions (these are not a part of homework): 2.5.7, 2.5.8, 2.5.19, 2.5.23, 2.5.24, 2.5.32, 2.5.38, 2.5.39, 2.5.40, 2.5.42, 2.5.45, 2.5.46

References