PROBLEM SET 5 (DUE: MAR 6, WED)

1. Do Exercise 2.5.10. in [OS18].

$Cx = 0 \Rightarrow \binom{A}{B}x = 0 \Rightarrow \binom{Ax}{Bx} = 0 \Rightarrow Ax = 0$, thus $\ker C \subset \ker A$. Also if $A = (0)$ and $C = \binom{0}{1}$, then $\ker C = \{(0)\} \neq \mathbb{R} = \ker A$.

2. Do Exercise 2.5.11. in [OS18].

It is clear since $\mathrm{im}C = \mathrm{span}\{\text{columns of } C\} \supset \mathrm{span}\{\text{columns of } A\} = \mathrm{im}A$. Also if $A = (0)$ and $C = (0\ 1)$ then $\mathrm{im}C = \mathbb{R} \neq \{(0)\} = \mathrm{im}A$.

3. Do Exercise 2.5.14. in [OS18].

(a) follows from direct calculation. For (b), note that the columns are not all parallel to one another, thus $\mathrm{rank}A \geq 2$ which means $\dim\ker A \leq 1$ by the fundamental theorem of linear algebra.. As $x_1^* - x_2^* = (4, -2, 2)^T$ satisfies $A(x_1^* - x_2^*) = 0$, it follows that $(4, -2, 2)^T \in \ker A$. Thus we conclude that $\dim\ker A = 1$ and $\ker A = \mathrm{span}\{(4, -2, 2)^T\}$. It means that the general solution is given by $\ker A + x_1^* = \{t(4, -2, 2)^T + (1, 1, 0)^T \mid t \in \mathbb{R}\}$.

4. Do Exercise 2.5.21. in [OS18].

(a) (i) $\mathrm{span}\{(1, 2)^T\}$ (ii) $\mathrm{span}\{(1, -3)^T\}$ (iii) $\mathrm{span}\{(3, 1)^T\}$ (iv) $\mathrm{span}\{(-2, 1)^T\}$

(b) (i) $\mathrm{span}\{(1, 0, -1)^T, (0, 1, 2)^T\}$ (ii) $\mathrm{span}\{(1, 2, 0)^T, (0, 0, 1)^T\}$
(iii) $\mathrm{span}\{(-2, 1, 0)^T\}$ (iv) $\mathrm{span}\{(1, -2, 1)^T\}$

(c) (i) $\mathrm{span}\{(1, 0, 3)^T, (0, 1, -1)^T\}$ (ii) $\mathrm{span}\{(1, 0, -1, 3)^T, (0, 1, 3, -2)^T\}$
(iii) $\mathrm{span}\{(-3, 2, 0, 1)^T, (1, -3, 1, 0)^T\}$ (iv) $\mathrm{span}\{(-3, 1, 1)^T\}$

(d) (i) $\mathrm{span}\{(1, 0, 2, 0, -2)^T, (0, 1, 1, 0, -1)^T, (0, 0, 0, 1, 1)^T\}$
(ii) $\mathrm{span}\{(1, 0, -4, 2, 0)^T, (0, 1, -2, 0, 0)^T, (0, 0, 0, 0, 1)^T\}$
(iii) $\mathrm{span}\{(-2, 0, 0, 1, 0)^T, (4, 2, 1, 0, 0)^T\}$
(iv) $\mathrm{span}\{(2, 1, 0, -1, 1)^T, (-2, -1, 1, 0, 0)^T\}$

5. Do Exercise 2.5.30. in [OS18].

(a) It is obvious as $A = A^T$.

(b) $\text{im}A = \text{coim}A = \text{span}\{(1, 0, -2)^T, (0, 1, 1)^T\}$, $\ker A = \text{coker}A = \text{span}\{(2, -1, 1)^T\}$.

(c) No. If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $A^T = -A$, thus $\text{coker}A = \ker A^T = \ker(-A) = \ker A$ and $\text{coim}A = \text{im}A^T = \text{im}(-A) = \text{im}A$. However A is not symmetric.

— More Exercises Suggestions (these are not a part of homework): 2.5.7, 2.5.8, 2.5.19, 2.5.23, 2.5.24, 2.5.32, 2.5.38, 2.5.39, 2.5.40, 2.5.42, 2.5.45, 2.5.46

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.