PROBLEM SET 6 (DUE: MAR 13, WED)

1. Do Exercise 3.1.3 in [OS18].

$\langle (1,-1),(1,-1)\rangle = 0$, thus it does not satisfy positivity.

2. Do Exercise 3.1.12. in [OS18].

(a) We have

$$\begin{aligned}||u+v||^2 - ||u-v||^2 &= \langle u+v, u+v\rangle - \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle - \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle - \langle v,v\rangle \\ &= 4\langle u,v\rangle,\end{aligned}$$

thus $\langle u,v\rangle = (||u+v||^2 - ||u-v||^2)/4$ as desired.

(b) For $u = (u_1,u_2)^T$ and $v = (v_1,v_2)^T$, we have

$$\begin{aligned}\langle u,v\rangle &= (||u+v||^2 - ||u-v||^2)/4 \\ &= \frac{1}{4}\big((u_1+v_1)^2 - 3(u_1+v_1)(u_2+v_2) + 5(u_2+v_2)^2) \\ &\qquad -((u_1-v_1)^2 - 3(u_1-v_1)(u_2-v_2) + 5(u_2-v_2)^2)\big) \\ &= \frac{1}{4}(4u_1v_1 - 6(u_1v_2 + v_1u_2) + 20u_2v_2) \\ &= u_1v_1 - \frac{3}{2}u_1v_2 - \frac{3}{2}v_1u_2 + 5u_2v_2.\end{aligned}$$

3. Do Exercise 3.1.13. in [OS18].

(a) We have

$$\begin{aligned}||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\ &= 2\langle x,x\rangle + 2\langle y,y\rangle = 2(||x||^2 + ||y||^2),\end{aligned}$$

which is what we want to prove.

(b) This is the parallelogram law applied to the parallelogram with sides x and y, see e.g. `https://en.wikipedia.org/wiki/Parallelogram_law`.

4. Do Exercise 3.2.18. in [OS18].

Let $A = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{smallmatrix}\right)$. Then the vectors orthogonal to $(1,2,3,4)^T$ and $(5,6,7,8)^T$ are exactly the vectors in the kernel of $\ker A$. By direct calculation, we see that they are given by $s(1,-2,1,0)^T + t(2,-3,0,1)^T$ for $s, t \in \mathbb{R}$.

5. Do Exercise 3.2.28. in [OS18].

Let $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$ such that $a \neq 0$. Then $\langle f, e^x \rangle = 0$ if and only if

$$\int_0^1 f e^x dx = a(e-2) + b + c(e-1) = 0$$

Thus $f(x) = ax^2 + (-a(e-2) - c(e-1))x + c$ for some $a, c \in \mathbb{R}$, $a \neq 0$.

— More Exercises Suggestions (these are not a part of homework): 3.1.1, 3.1.5, 3.1.6, 3.1.10, 3.1.14, 3.1.19, 3.1.20, 3.1.23, 3.1.25, 3.1.27, 3.1.28, 3.1.31, 3.1.33, 3.2.3, 3.2.6, 3.2.7, 3.2.9, 3.2.10, 3.2.16, 3.2.21, 3.2.25, 3.2.26, 3.2.29, 3.2.30, 3.2.31, 3.2.36, 3.2.37, 3.2.38, 3.2.39, 3.2.40, 3.2.41

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.