PROBLEM SET 7 (DUE: MAR 27, WED)

1. Do Exercise 3.3.14 in [OS18].

By Exercise 3.1.13, it suffices to show that the ∞ norm does not satisfy the equation $||x+y||^2+||x-y||^2 = 2(||x||^2+||y||^2)$. Now if we set $x = (1,0)^T$ and $y = (0,2)^T$, then we have $||x+y||_\infty = ||x-y||_\infty = ||y||_\infty = 2$ and $||x||_\infty = 1$, but $2^2+2^2 \neq 2(1^2+2^2)$.

2. Do Exercise 3.3.47. in [OS18].

It is false. If $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, then $B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, but $||B||_\infty = 2 \neq 1 = ||A||_\infty$.

3. Do Exercise 3.4.8. in [OS18].

Let $K = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then they are both positive definite (one may use Sylvester's criterion), but $KL = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$ is not even symmetric.

4. Do Exercise 3.4.25. in [OS18].

We have

$$\left\langle e^{ax}, e^{bx} \right\rangle = \int_0^1 e^{ax}e^{bx}dx = \begin{cases} \left.\dfrac{e^{(a+b)x}}{a+b}\right|_0^1 = \dfrac{e^{a+b}-1}{a+b} & \text{if } a+b \neq 0, \\ x|_0^1 = 1 & \text{if } a+b = 0. \end{cases}$$

Thus the associated Gram matrix K is given by

$$K = \begin{pmatrix} 1 & e-1 & \frac{e^2-1}{2} \\ e-1 & \frac{e^2-1}{2} & \frac{e^3-1}{3} \\ \frac{e^2-1}{2} & \frac{e^3-1}{3} & \frac{e^4-1}{4} \end{pmatrix}.$$

This matrix is positive definite because $\{1, e^x, e^{2x}\}$ is linearly independent.

5. Do Exercise 3.5.8. in [OS18].

If we define a symmetric matrix

$$K = \begin{pmatrix} 1 & \frac{a}{2} & \frac{b}{2} \\ \frac{a}{2} & 1 & \frac{c}{2} \\ \frac{b}{2} & \frac{c}{2} & 1 \end{pmatrix},$$

then by direct calculation the given quadratic form is equal to $(x, y, z)K(x, y, z)^T$. Thus this quadratic form is positive definite if and only if K is positive definite. But we have $K = LDL^T$ where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a}{2} & 1 & 0 \\ \frac{b}{2} & \frac{ab-2c}{a^2-4} & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{a^2}{4} & 0 \\ 0 & 0 & \frac{a^2-abc+b^2+c^2-4}{a^2-4} \end{pmatrix},$$

thus K is positive definite if and only if $1 - \frac{a^2}{4} > 0$ and $\frac{a^2-abc+b^2+c^2-4}{a^2-4} > 0$, i.e.

$$-2 < a < 2 \text{ and } a^2 + b^2 + c^2 - abc - 4 < 0.$$

6. Do Exercise 3.6.27. in [OS18].

It is false, since it is not stable under multiplication by i. Indeed, we have

$$i\overline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = i\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ i \end{pmatrix} \neq \overline{\begin{pmatrix} i \\ i \end{pmatrix}} = \begin{pmatrix} i \\ -i \end{pmatrix}.$$

7. Do Exercise 3.6.40. in [OS18].

(a) We have

$$\begin{aligned} ||z+w||^2 &= \langle z+w, z+w \rangle = \langle z, z \rangle + \langle z, w \rangle + \langle w, z \rangle + \langle w, w \rangle \\ &= ||z||^2 + ||w||^2 + \langle z, w \rangle + \overline{\langle z, w \rangle} \\ &= ||z||^2 + ||w||^2 + 2\Re \langle z, w \rangle. \end{aligned}$$

(b) Using part (a), we have

$$\begin{aligned} &||z+w||^2 - ||z-w||^2 + i||z+iw||^2 - i||z-iw||^2 \\ &= 2\Re\langle z, w \rangle - 2\Re\langle z, -w \rangle + 2i\Re\langle z, iw \rangle - 2i\Re\langle z, -iw \rangle \\ &= 2\Re\langle z, w \rangle + 2\Re\langle z, w \rangle + 2i\Re(-i\langle z, w \rangle) - 2i\Re(i\langle z, w \rangle) \\ &= 4(\Re\langle z, w \rangle + i\Re(-i\langle z, w \rangle)), \end{aligned}$$

but the last one equals $4\langle z, w \rangle$. Indeed, for any $a, b \in \mathbb{R}$ we have

$$\Re(a+bi) + i\Re(-i(a+bi)) = \Re(a+bi) + i\Re(b-ai) = a+bi,$$

thus $\Re x + i\Re(-ix) = x$ for any $x \in \mathbb{C}$. From this the claim follows.

— More Exercises Suggestions (these are not a part of homework): 3.3.1, 3.3.4, 3.3.7, 3.3.9, 3.3.13, 3.3.16, 3.3.19, 3.3.23, 3.3.27, 3.3.36, 3.3.41, 3.3.43, 3.3.51, 3.3.52, 3.4.2, 3.4.5, 3.4.10, 3.4.14, 3.4.20, 3.4.22, 3.4.26, 3.4.29, 3.4.30, 3.4.35, 3.5.2, 3.5.6, 3.5.10, 3.5.13, 3.5.15, 3.5.19, 3.5.21, 3.6.1 (answer: Euler's identity), 3.6.26, 3.6.30, 3.6.36, 3.6.43, 3.6.44, 3.6.45, 3.6.46, 3.6.49

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.