PROBLEM SET 8 (DUE: APR 3, WED)

1. Do Exercise 4.1.4. in [OS18].

> Since $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$ and $\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle e_3, e_1\rangle = 0$ by direct calculation, it is an orthogonal basis. Also, after normalization we obtain an orthonormal basis $\{\frac{e_1}{\|e_1\|}, \frac{e_2}{\|e_2\|}, \frac{e_3}{\|e_3\|}\} = \{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}\}$.

2. Do Exercise 4.1.9. in [OS18].

> It is false. For $\langle v, w\rangle = av_1w_1 + bv_2w_2 + cv_3w_3$ to be an inner product, we should have $a, b, c > 0$. However, if we choose a basis $(1,0,0)^T, (0,1,0)^T, (0,1,1)^T$ of $\mathbb{R}^3$, then $\langle (0,1,0)^T, (0,1,1)^T\rangle = b \neq 0$, thus $(0,1,0)^T$ and $(0,1,1)^T$ are not orthogonal.

3. Do Exercise 4.1.13. in [OS18].

> (a) It follows since the (i,j)-entry of $A^TKA = I$ equals $\langle v_i, v_j\rangle = \delta_{ij}$. (Here, δ_{ij} is 1 if $i = j$ and 0 otherwise.)
>
> (b) Set A as in (a) and let $K = (A^T)^{-1}A^{-1}$. Then $A^TKA = I$, thus the column vectors of A are an orthonormal basis with respect to the inner product defined by K. Furthermore, this inner product is uniquely determined as $K = (A^T)^{-1}A^{-1}$ is the only matrix satisfying $A^TKA = I$.
>
> (c) We calculate $K = (A^T)^{-1}A^{-1}$ when $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ to get $K = \begin{pmatrix} 10 & -7 \\ -7 & 5 \end{pmatrix}$. Thus the inner product is defined by $\langle v, w\rangle = v^T \begin{pmatrix} 10 & -7 \\ -7 & 5 \end{pmatrix} w$.
>
> (d) We calculate $K = (A^T)^{-1}A^{-1}$ when $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ to get $K = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 6 & -3 \\ 0 & -3 & 2 \end{pmatrix}$.
>
> Thus the inner product is defined by $\langle v, w\rangle = v^T \begin{pmatrix} 3 & -2 & 0 \\ -2 & 6 & -3 \\ 0 & -3 & 2 \end{pmatrix} w$.

4. Do Exercise 4.2.3. in [OS18].

When performing the Gram-Schmidt process, one can see that u_3 equals 0. This is because $v_3 \in \operatorname{span}\{v_1, v_2\}$.

5. Do Exercise 4.2.5. in [OS18].

We perform the Gram-Schmidt process to have

$$\begin{aligned}
u_1 &= w_1 = (1,-1,-1,1,1)^T \\
u_2 &= w_2 - \frac{\langle w_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = (2,1,4,-4,2)^T + (1,-1,-1,1,1)^T = (3,0,3,-3,3)^T \\
u_3 &= w_3 - \frac{\langle w_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle w_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2 \\
&= (5,-4,-3,7,1)^T - 4(1,-1,-1,1,1)^T + \frac{1}{3}(3,0,3,-3,3)^T = (2,0,2,2,-2)^T
\end{aligned}$$

Thus after normalization, $\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\}$ equals

$$\{(\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})^T, (\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^T, (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^T$$

6. Do Exercise 4.2.8. in [OS18].

(a) Note that $(1,0)^T$ and $(0,1)^T$ are orthogonal. Thus after normalization we have $\{(\frac{1}{\sqrt{3}}, 0)^T, (0, \frac{1}{\sqrt{5}})^T\}$.

(b) Note that $(1,0)^T$ and $(1,4)^T$ are orthogonal. Thus after normalization we have $\{(\frac{1}{2}, 0)^T, (\frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{3}})^T\}$.

(c) Note that $(1,0)^T$ and $(1,-2)^T$ are orthogonal. Thus after normalization we have $\{(\frac{1}{\sqrt{2}}, 0)^T, (\frac{1}{\sqrt{10}}, \frac{-2}{\sqrt{10}})^T\}$.

— More Exercises Suggestions (these are not a part of homework): 4.1.1, 4.1.2, 4.1.3, 4.1.6, 4.1.10, 4.1.12, 4.1.15, 4.1.16, 4.1.18, 4.1.24, 4.1.26, 4.1.28, 4.1.31, 4.1.32, 4.2.1, 4.2.2, 4.2.6, 4.2.7, 4.2.9, 4.2.12, 4.2.13, 4.2.14, 4.2.20, 4.2.21

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.