PROBLEM SET 9 (DUE: APR 10, WED)

1. Do Exercise 4.3.11. in [OS18].

Suppose that Q is the diagonal orthogonal matrix with $d_1, d_2, \ldots, d_n$ on the diagonal. Then $Q^TQ = Q^2 = I$, thus $d_1^2 = d_2^2 = \ldots = d_n^2 = 1$, i.e $d_1, d_2, \ldots, d_n \in \{1, -1\}$. In other words, Q is a diagonal matrix with ± 1 on the diagonal.

2. Do Exercise 4.3.16. in [OS18].

(a) It follows from that $\|x\|^2 = x^Tx = x^TIx = x^TQ^TQx = (Qx)^T(Qx) = \|Qx\|^2$.

(b) Note that we have

$$\|x+y\|^2 - \|x\|^2 - \|y\|^2 = (x+y)^T(x+y) - x^Tx - y^Ty = 2x^Ty$$

by direct calculation. Therefore, we have

$$\begin{aligned} 2x^TQ^TQy = 2(Qx)^t(Qy) &= \|Qx+Qy\|^2 - \|Qx\|^2 - \|Qy\|^2 \\ &= \|Q(x+y)\|^2 - \|Qx\|^2 - \|Qy\|^2 \\ &= \|x+y\|^2 - \|x\|^2 - \|y\|^2 = 2x^Ty, \end{aligned}$$

i.e. $x^TQ^TQy = x^Ty$ for any $x, y \in \mathbb{R}^n$. Now letting x, y be standard basis elements e_i, e_j gives that the (i,j)-entry of Q^TQ is equal to 1 if $i = j$ and 0 otherwise, but it means that $Q^TQ = I$, thus Q is orthogonal. (For another proof, see the solution of Problem 6 on the second midterm exam.)

3. Do Exercise 4.3.19. in [OS18].

If A and B are the matrices with v_i and w_i as their column vectors respectively, then direct calculation shows that $BA^{-1}v_i = w_i$ for any i. Thus there always exists a matrix Q such that $Qv_i = w_i$ for any i. Now we have

$$\begin{aligned} Q \text{ is orthogonal} \Leftrightarrow Q^TQ = I &\Leftrightarrow v_i^TQ^TQv_j = v_i^Tv_j \text{ for all } i, j \\ &\Leftrightarrow w_i^Tw_j = v_i^Tv_j \text{ for all } i, j \end{aligned}$$

from which the assertion follows.

4. Do Exercise 4.3.26. in [OS18].

As given in Example 4.17, we have

$$Q = \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ -\frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3}. \end{pmatrix}$$

Thus we also have

$$R = Q^T A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 2\sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{3}{\sqrt{2}} \end{pmatrix}.$$

5. Do Exercise 4.3.28. in [OS18].

(a) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{5}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, thus

$$\begin{pmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{5}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore y = \frac{1}{5}, x = -\frac{7}{5}$$

(b) $\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 & 2 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, thus

$$\begin{pmatrix} 3 & 0 & 2 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}^T \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

$$\therefore z = -1, y = -1, x = 1$$

(c) $$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ thus}$$

$$\begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\therefore z = \frac{1}{2}, y = \frac{1}{2}, x = -\frac{1}{2}$$

6. Do Exercise 4.3.34.(a) in [OS18].

(i) $I - 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $I - 2\begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix} = \begin{pmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{pmatrix}$

(iii) $I - 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iv) $I - 2\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

— More Exercises Suggestions (these are not a part of homework): 4.3.3, 4.3.4, 4.3.10, 4.3.14, 4.3.17, 4.3.18, 4.3.23, 4.3.32, 4.3.34

References

[OS18] Olver, P. J. and Shakiban, C., *Applied Linear Algebra*, 2nd ed., Undergraduate Texts in Mathematics, Springer, 2018.