Math 4242 1st Mid-Term   Name: ______________________

Question 1. True of False .... 20 points

Mark each of the following "T" (if true) of "F" (if false).

2 pts (a) **T** A singular matrix cannot be regular.

3 pts (b) **F** If A and B are symmetric $n \times n$ matrices, then so is AB.

3 pts (c) **T** If $\operatorname{coker} A = \operatorname{coker} B$, then $\operatorname{rank} A = \operatorname{rank} B$.

3 pts (d) **T** If $\{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^m$ is linearly independent and $E \in \mathrm{Mat}_{m,m}(\mathbb{R})$ is nonsingular, then $\{Ev_1, Ev_2, \dots, Ev_n\} \subset \mathbb{R}^m$ is also linearly independent.

3 pts (e) **T** A permuted LU factorization of a nonsingular matrix A can be unique for some A.

3 pts (f) **F** Let $f(x, y, z)$ be a nonzero real polynomial in variables x, y, and z. Then the set $\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$ is a real vector subspace of $\mathbb{R}^3$ if and only if the degree of f is 1 and there is no constant term in f, i.e. if f is homogeneous of degree 1.

3 pts (g) **F** $\{(A, v) \in \mathrm{Mat}_{n,n}(\mathbb{R}) \times \mathbb{R}^n \mid Av = 0\}$ is a vector subspaces of $\mathrm{Mat}_{n,n}(\mathbb{R}) \times \mathbb{R}^n$.

Question 2. ***LU*** **factorization . . . . 20 points**

Consider the following matrix.

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$$

8 pts (a) Find the LU factorization of A.

8 pts (b) Find the solution of $Ax = (14, 7, 7, 14)^T$.

4 pts (c) Find the determinant of A.

Answer. (a) We can find U by the following process.

$$\begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 3 & 0 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & -6 & -17 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -\frac{49}{5} \end{pmatrix} = U$$

Since $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -\frac{49}{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{6}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and A is symmetric, we have

$$L = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{6}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 3 & \frac{6}{5} & 1 \end{pmatrix}.$$

(b) If we set $y \in \mathbb{R}^4$ to be $Ly = (14, 7, 7, 14)^T$, then using forward substitution we have $y = (14, 21, -35, -49)^T$. Now the solution $x \in \mathbb{R}^4$ satisfying $Ax = (14, 7, 7, 14)^T$ is given by using back substitution on $Ux = y$, and we have $x = (3, 4, 1, 5)^T$.

(c) $\det A = \det L \det U = \det U = 49$.

Question 3. Row Echelon Form . . . . 20 points

Consider the following matrix.

$$A = \begin{pmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{pmatrix}$$

5 pts (a) Find a row echelon form of A. (It does not need to be reduced.)

7 pts (b) Find the rank of A, and a basis of each of $\ker A, \operatorname{im} A$, and $\operatorname{coim} A$.

4 pts (c) By direct calculation, one can show that $(10, -5, 4, 1)A = 0$. From this, explain why $\{(10, -5, 4, 1)^T\}$ is a basis of $\operatorname{coker} A$.

4 pts (d) Find a row echolon form of A^T. (It does not need to be reduced.)

Answer. (a) We have

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 2 & -8 & -5 & 0 \\ 0 & -3 & 12 & 5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -4 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the last one is a row echelon form of A.

(b) Let us call the matrix obtained in (a) R. The rank of A is 3 as there are three nonzero pivots. Also, as $\operatorname{coim} A = \operatorname{coim} R$, we see that

$$\{(1, 0, -3, 1, 2)^T, (0, 1, -4, -3, 1)^T, (0, 0, 0, 1, -2)^T\}$$

is a basis of $\operatorname{coim} A$. For the image, we pick the columns of A corresponding to the nonzero pivots of R, thus $\{(1, 0, -3, 2)^T, (0, 1, 2, 3)^T, (1, -3, -8, 7)^T\}$ is a basis of $\operatorname{im} A$. For the kernel, we use back substitution to solve the homogeneous linear system $Ax = 0$, and see that $\{(-4, 5, 0, 2, 1)^T, (3, 4, 1, 0, 0)^T\}$ is a basis of $\ker A$.

(c) We only need to show that $\operatorname{coker} A$ is 1 dimensional, but it is clear as $\dim \operatorname{coker} A = 4 - \operatorname{rank} A = 1$ by the fundamental theorem of linear algebra.

(d) We can take e.g. $\begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. (Note that this is the only row reduced echelon form of size 5×4 whose kernel is equal to $\ker A^T = \operatorname{span}\{(10, -5, 4, 1)^T\}$.)

Question 4. More Row Echelon Form . . . . 20 points

Consider the following matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$$

5 pts (a) Find a row echelon form of A. (It does not need to be reduced.)

7 pts (b) Find the rank of A, and a basis of each of $\ker A, \operatorname{im} A$, and $\operatorname{coim} A$.

4 pts (c) By direct calculation, one can show that $(-3, 1, 1, 0)A = 0$. Find another vector $y \in \mathbb{R}^4$ such that $\{y, (-3, 1, 1, 0)^T\}$ becomes a basis of $\operatorname{coker} A$.

4 pts (d) Find two polynomials $f(x, y, z), g(x, y, z)$ such that $f(a, b, c) = g(a, b, c) = 0$ if and only if $(a, b, c, 0)^T \in \operatorname{im} A$.

Answer. (a) We have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the last one is a row echelon form of A.

(b) Let us call the matrix obtained in (a) R. The rank of A is 2 as there are two nonzero pivots. Also, as $\operatorname{coim} A = \operatorname{coim} R$, we see that

$$\{(1, 2, 3, 4, 5)^T, (0, -1, -2, -3, -4)^T\}$$

is a basis of $\operatorname{coim} A$. For the image, we pick the columns of A corresponding to the nonzero pivots of R, thus $\{(1, 2, 1, 2)^T, (2, 3, 3, 4)^T\}$ is a basis of $\operatorname{im} A$. For the kernel, we use back substitution to solve the homogeneous linear system $Ax = 0$, and see that $\{(1, -2, 1, 0, 0)^T, (2, -3, 0, 1, 0)^T, (3, -4, 0, 0, 1)^T\}$ is a basis of $\ker A$.

(c) One can take e.g. $y = (2, 0, 0, -1)^T$. (Note that the fourth row of A is twice the first row.)

(d) Since $\operatorname{im} A$ is spanned by $(1, 2, 1, 2)^T, (2, 3, 3, 4)^T$, we may write any $v = (a, b, c, 0)^T \in \operatorname{im} A$ as a linear combination of them, say $s(1, 2, 1, 2)^T + t(2, 3, 3, 4)^T = (s + 2t, 2s + 3t, s + 3t, 2s + 4t)$ for some $s, t \in \mathbb{R}$. As the fourth coordinate of v is zero, we have $2s+4t = 0$, i.e. $s = -2t$, thus $v = (0, -t, t, 0)$ for some $t \in \mathbb{R}$, i.e. $a = 0, b = -t, c = t$ for some $t \in \mathbb{R}$. Thus we may take e.g. $f(x, y, z) = x$ and $g(x, y, z) = y + z$.

Question 5. Cartan Matrix of Type A .... 20 points

Let A_n be the matrix of size $n \times n$ whose (i, j)-entry a_{ij} is defined to be

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, we have $A_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$.

5 pts (a) Find and justify the formula for $\det A_n$. (Partial credit will be given for guessing the correct formula.)

5 pts (b) Find a basis of each of $\ker A_n, \operatorname{im} A_n, \operatorname{coker} A_n$, and $\operatorname{coim} A_n$.

Now we consider the matrix B_n of size $n \times n$ whose (i, j)-entry b_{ij} is defined to be

$$b_{ij} = \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, we have $B_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

5 pts (c) Prove that $\det B_n = 0$ when n is odd.

5 pts (d) Prove that $\det B_n \in \{1, -1\}$ when n is even.

Answer. (a) Using row operation, we see that the matrix U in the LU factorization of A_n is given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots \\ 0 & \frac{3}{2} & -1 & 0 & \cdots \\ 0 & 0 & \frac{4}{3} & -1 & \cdots \\ 0 & 0 & 0 & \frac{5}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the pivots are $2, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+1}{n}$. Thus we have $\det A = \det U = \prod_{i=1}^{n} \frac{i+1}{i} = n + 1$.

(b) Since $\det A_n = n + 1 \neq 0$ for any $n \geq 1$, A_n is always nonsingular. Thus $\ker A_n = \operatorname{coker} A_n = \{0\}$ and $\operatorname{im} A_n = \operatorname{coim} A_n = \mathbb{R}^n$. In other words, $\{\}$ is a basis of both $\ker A_n$ and $\operatorname{coker} A_n$, and $\{e_1, e_2, \dots, e_n\}$ (standard basis) is a basis of both $\operatorname{im} A_n$ and $\operatorname{coim} A_n$.

(c) Let us set $v = (1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0, (-1)^{\frac{n-1}{2}})^T$. Then by direct calculation, we have $B_n v = 0$, i.e. $\dim\ker B_n \geq 1$. Thus $\det B_n = 0$.

(d) We successively perform the row operation

$$[n-2] \leftarrow [n-2]-[n], \quad [n-4] \leftarrow [n-4]-[n-2], \quad \ldots, \quad [6] \leftarrow [6]-[4], \quad [4] \leftarrow [4]-[2]$$

and

$$[3] \leftarrow [3]-[1], \quad [5] \leftarrow [5]-[3], \quad \ldots, \quad [n-3] \leftarrow [n-3]-[n-5], \quad [n-1] \leftarrow [n-1]-[n-3].$$

(Note that here we only used the row operation of the first kind, which implies that the determinant is not changed.) Then the result is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ i.e. a “block-diagonal” matrix consisting of } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, this is a permutation matrix. However, in general any permutation matrix P satisfies $PP^T = I$, from which we have $(\det P)^2 = \det P \det P^T = \det(PP^T) = \det I = 1$, thus $\det P = \pm 1$. Thus we conclude that $\det B_n = \pm 1$.