Question 1. True or False ............................................ 10 points

Mark each of the following “T” (if true) of “F” (if false).

2 pts  (a)  **T**  A symmetric positive-definite matrix is regular.

2 pts  (b)  **T**  If \( Q \) is an orthogonal matrix, then so is \( Q^n \) for any \( n \in \{1, 2, 3, \ldots \} \).

2 pts  (c)  **T**  There exist \( v_1, v_2 \in \mathbb{R}^2 \) such that
\[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}
\]
is their associated Gram matrix with respect to the usual dot product on \( \mathbb{R}^2 \).

2 pts  (d)  **F**  If \( A, B, S \in \text{Mat}_{2 \times 2}(\mathbb{R}) \) satisfy
\[
B = S^{-1}AS,
\]
then \( \|B\|_\infty = \|A\|_\infty \). Here, \( \| \cdot \|_\infty : \text{Mat}_{2 \times 2}(\mathbb{R}) \to \mathbb{R} \) is the matrix norm associated with the \( \infty \)-norm on \( \mathbb{R}^2 \).

2 pts  (e)  **F**  The set of complex vectors \( \left\{ \begin{pmatrix} a + bi \\ a \end{pmatrix} \in \mathbb{C}^2 : a, b \in \mathbb{R}^2 \right\} \) is a complex vector subspace of \( \mathbb{C}^2 \). (Here, \( i \) is a square root of -1.)
Question 2. Inner Product and Parallelogram Law ................. 20 points

(a) Let $\langle \cdot, \cdot \rangle$ be an inner product on a real vector space $V$ and let $\| \cdot \|$ be the norm on $V$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in V$. Prove the following “parallelogram law”: for any $x, y \in V$, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(b) Prove that the function $\| \cdot \|_1 : \mathbb{R}^2 \to \mathbb{R} : (a, b)^T \mapsto |a| + |b|$ is a norm on $\mathbb{R}^2$.

(c) Prove that there does not exist any inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ such that $\|x\|_1 = \sqrt{\langle x, x \rangle}$ for all $x \in \mathbb{R}^2$.

Answer.

(a) We have

$$\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y \rangle + \langle x - y, x - y \rangle$$
$$= \langle x, x \rangle + 2 \langle x, y \rangle + \langle y, y \rangle + \langle x, x \rangle - 2 \langle x, y \rangle + \langle y, y \rangle$$
$$= 2 \langle x, x \rangle + 2 \langle y, y \rangle = 2(\|x\|^2 + \|y\|^2).$$

(b) It satisfies

i) Positivity: $\|(a, b)^T\|_1 = |a| + |b| \geq 0$ and $\|(a, b)^T\|_1 = 0$ if and only if $a = b = 0$ if and only if $(a, b)^T = 0$.

ii) Homogeneity: $\|c(a, b)^T\|_1 = \|(ca, cb)^T\|_1 = |ca| + |cb| = |c|(|a| + |b|) = |c|\|(a, b)^T\|_1$ for any $c \in \mathbb{R}$.

iii) Triangle Inequality: $\|(a, b)^T + (c, d)^T\|_1 = |a + c| + |b + d| \geq |a| + |c| + |b| + |d| = \|(a, b)^T\|_1 + \|(c, d)^T\|_1$.

Thus $\| \cdot \|_1$ is a norm on $\mathbb{R}^2$.

(c) If $\| \cdot \|_1$ is associated with some inner product on $\mathbb{R}^2$, then it should satisfy the equation in part (a). However, if we set $x = (1, 0)^T$ and $y = (0, 2)^T$ then direct calculation shows that $\|x + y\|_1 = \|x - y\|_1 = 3$, $\|x\|_1 = 1$, and $\|y\|_1 = 2$ which does not satisfy the equation above since $3^2 + 3^2 = 18 \neq 10 = 2(1^2 + 2^2)$. 

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Answer. The row-reduced echelon form of $A$ is given by 
\[
\begin{pmatrix}
1 & 0 & -1 & -2 & -3 \\
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(a) $v_1 = (1, 2, 1, 2)^T, v_2 = (2, 3, 3, 4)^T$ are a basis of $\text{im } A$. By applying Gram-Schmidt, we have 
\[
\begin{align*}
    u_1 &= v_1 = (1, 2, 1, 2)^T \\
    u_2 &= v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = (2, 3, 3, 4)^T - \frac{19}{10} (1, 2, 1, 2)^T = \left( 1, \frac{11}{5}, \frac{1}{5}, \frac{2}{5} \right)^T
\end{align*}
\]

(b) $v_1 = (1, -2, 1, 0, 0)^T, v_2 = (2, -3, 0, 1, 0)^T, v_3 = (3, -4, 0, 0, 1)^T$ are a basis of $\text{ker } A$. By applying Gram-Schmidt, we have 
\[
\begin{align*}
    u_1 &= v_1 = (1, -2, 1, 0, 0)^T \\
    u_2 &= v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = (2, -3, 0, 1, 0)^T - \frac{1}{3} (1, -2, 1, 0, 0)^T = \left( \frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, 1, 0 \right)^T \\
    u_3 &= v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 \\
    &= (3, -4, 0, 0, 1)^T - \frac{11}{6} (1, -2, 1, 0, 0)^T = \left( \frac{1}{2}, 0, -\frac{1}{2}, -1, 1 \right)^T
\end{align*}
\]
Question 4. QR Factorization ................................. 15 points

Consider the following matrix.

\[ A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 6 \end{pmatrix} \]

(a) Find the QR factorization of \( A \).

(b) Find the QR factorization of \( A^T \).

Answer.

(a) Let us apply Gram-Schmidt to the column vectors of \( A \).

\[ u_1 = v_1 = (2, 1, 1)^T \]
\[ u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = (0, 1, -1)^T \]
\[ u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 \]
\[ = (0, 0, 6)^T - (2, 1, 1)^T + 3(0, 1, -1)^T = (-2, 2, 2)^T \]

\[ \therefore Q = \begin{pmatrix} u_1^T/\|u_1\| & u_2^T/\|u_2\| & u_3^T/\|u_3\| \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{6}} & 0 & -\sqrt{\frac{1}{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{3}} \\ \frac{\sqrt{3}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \]

\[ R = Q^T A = \begin{pmatrix} \sqrt{\frac{2}{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{6}{3}} & 0 & \sqrt{\frac{6}{3}} \\ 0 & \sqrt{\frac{2}{3}} & -3\sqrt{\frac{2}{3}} \\ 0 & 0 & 2\sqrt{\frac{3}{3}} \end{pmatrix} \]

(b) \( A^T \) is already upper-triangular with positive diagonal entries, i.e. we may set \( Q = I \) and \( R = A^T \).
Question 5. Orthonormal Bases ........................................... 15 points

Find an orthonormal basis of \( \mathbb{R}^3 \) for each inner product.

\[ \langle x, y \rangle = x^T \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{pmatrix} y. \]

\[ \langle x, y \rangle = x^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} y. \]

\[ \text{Answer.} \]

(a) The given inner product is a weighted dot product. Thus, we may choose
\[ u_1 = (\frac{1}{2}, 0, 0)^T, \quad u_2 = (0, \frac{1}{3}, 0)^T, \quad \text{and} \quad u_3 = (0, 0, \frac{1}{5})^T. \]

(b) Let us apply Gram-Schmidt to \( v_1 = (1, 0, 0)^T, v_2 = (0, 0, 1)^T, \) and \( v_3 = (0, 1, 0)^T. \)

Then,
\[ u_1 = v_1 = (1, 0, 0)^T 
\]
\[ u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = (0, 0, 1)^T 
\]
\[ u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 
\]
\[ = (0, 1, 0)^T - \frac{1}{2}(1, 0, 0)^T - \frac{1}{2}(0, 0, 1)^T = (\frac{1}{2}, 1, \frac{1}{2})^T. \]

Thus after normalization we have
\[ \frac{u_1}{\|u_1\|} = (\frac{1}{\sqrt{2}}, 0, 0)^T, \quad \frac{u_2}{\|u_2\|} = (0, 0, \frac{1}{\sqrt{2}})^T, \quad \frac{u_3}{\|u_3\|} = (\frac{1}{2}, 1, \frac{1}{2})^T. \]
Let \( \| \| : \mathbb{R}^n \to \mathbb{R} \) be the standard Euclidean norm, i.e.
\[
\|(a_1, a_2, \ldots, a_n)^T\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.
\]

Recall its associated natural matrix norm
\[
\| \|_M : \text{Mat}_{n \times n}(\mathbb{R}) \to \mathbb{R} : A \mapsto \max \{ \| Au \| : \| u \| = 1 \}.
\]

(a) Prove the following statement: \( Q \in \text{Mat}_{n \times n}(\mathbb{R}) \) is an orthogonal matrix if and only if \( \| Qu \| = 1 \) for any unit vector \( u \in \mathbb{R}^n \).

(b) Prove that if \( Q \in \text{Mat}_{n \times n}(\mathbb{R}) \) is orthogonal then \( \| Q \|_M = 1 \).

(c) Find a matrix \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \) such that \( \| A \|_M = 1 \) but \( A \) is not orthogonal, and explain why your example satisfies the desired conditions. (Here, you may choose \( n \) to be any element in \( \{1, 2, 3, \ldots \} \) that you want.)

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**Answer.**

(a) If \( Q \) is an orthogonal matrix, then \( Q^T Q = I \) thus \( \| Qu \|^2 = u^T Q^T Q u = u^T u = 1 \) for any unit vector \( u \in \mathbb{R}^n \). Conversely, if \( \| Qu \|^2 = 1 \) for any unit vector \( u \in \mathbb{R}^n \) then \( \| Qx \|^2 = \| x \|^2 \) for any \( x \in \mathbb{R}^n \) by homogeneity, which means \( x^T Q^T Q x = x^T x \). Now we consider the following cases.

- If we set \( x = e_i \), then \( e_i^T Q^T Q e_i = e_i^T e_i = 1 \). It means that the diagonal entries of \( Q^T Q \) are all 1.

- If we set \( x = e_i + e_j \) for \( i \neq j \), then we have
  \[
  2 = (e_i + e_j)^T (e_i + e_j) = (e_i + e_j)^T Q^T Q (e_i + e_j)
  = e_i Q^T Q e_i + e_j Q^T Q e_j + e_i Q^T Q e_j + e_j Q^T Q e_i = 2 + e_i Q^T Q e_j
  \]
  since \( e_i Q^T Q e_i = e_i Q^T Q e_j = 1 \) and \( e_i Q^T Q e_j = (e_i Q^T Q e_j)^T = e_j Q^T Q e_i \).
  Thus \( e_i Q^T Q e_j = 0 \) and the \((i, j)\)-entry of \( Q^T Q \) is zero when \( i \neq j \).

  From this it follows that \( Q^T Q = I \) as desired.

(b) By part (a), we have \( \{ \| Qu \| : \| u \| = 1 \} = \{1\} \). Thus \( \| Q \|_M = \max \{1\} = 1 \).

(c) Set \( n = 2 \) and \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Then \( A \) is not orthogonal as \( A \) is not invertible.

  However, for any \( u = (a, b) \in \mathbb{R}^2 \) such that \( a^2 + b^2 = 1 \), we have \( \| Au \| = a^2 \).
  Thus, \( \{ \| Au \| : \| u \| = 1 \} = \{ a^2 : a^2 + b^2 = 1 \} = [0, 1] \). It follows that \( \| A \|_M = \max \{0, 1\} = 1 \).