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Question 1. True or False .... **10 points**

Mark each of the following "T" (if true) of "F" (if false).

2 pts (a) **T** A symmetric positive-definite matrix is regular.

2 pts (b) **T** If Q is an orthogonal matrix, then so is Q^n for any $n \in \{1, 2, 3, \dots\}$.

2 pts (c) **T** There exist $v_1, v_2 \in \mathbb{R}^2$ such that $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is their associated Gram matrix with respect to the usual dot product on $\mathbb{R}^2$.

2 pts (d) **F** If $A, B, S \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ satisfy $B = S^{-1}AS$, then $\|B\|_\infty = \|A\|_\infty$. Here, $\|\ \|_\infty : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ is the matrix norm associated with the ∞-norm on $\mathbb{R}^2$.

2 pts (e) **F** The set of complex vectors $\left\{ \begin{pmatrix} a + bi \\ a \end{pmatrix} \in \mathbb{C}^2 : a, b \in \mathbb{R}^2 \right\}$ is a complex vector subspace of $\mathbb{C}^2$. (Here, i is a square root of -1.)

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Question 2. Inner Product and Parallelogram Law .... 20 points

6 pts (a) Let $\langle\ ,\ \rangle$ be an inner product on a real vector space V and let $\|\ \|$ be the norm on V defined by $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in V$. Prove the following "parallelogram law": for any $x, y \in V$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

7 pts (b) Prove that the function $\|\ \|_1 : \mathbb{R}^2 \to \mathbb{R} : (a,b)^T \mapsto |a| + |b|$ is a norm on $\mathbb{R}^2$.

7 pts (c) Prove that there does not exist any inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^2$ such that $\|x\|_1 = \sqrt{\langle x, x\rangle}$ for all $x \in \mathbb{R}^2$.

Answer.

(a) We have

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle + \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle \\ &= 2\langle x, x\rangle + 2\langle y, y\rangle = 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

(b) It satisfies

i) Positivity: $\left\|(a,b)^T\right\|_1 = |a| + |b| \geq 0$ and $\left\|(a,b)^T\right\|_1 = 0$ if and only if $a = b = 0$ if and only if $(a,b)^T = 0$.

ii) Homogeneity: $\left\|c(a,b)^T\right\|_1 = \left\|(ca,cb)^T\right\|_1 = |ca| + |cb| = |c|(|a| + |b|) = |c|\left\|(a,b)^T\right\|_1$ for any $c \in \mathbb{R}$.

iii) Triangle Inequality: $\left\|(a,b)^T + (c,d)^T\right\|_1 = |a+c| + |b+d| \geq |a| + |c| + |b| + |d| = \left\|(a,b)^T\right\|_1 + \left\|(c,d)^T\right\|_1$.

Thus $\|\ \|_1$ is a norm on $\mathbb{R}^2$.

(c) If $\|\ \|_1$ is associated with some inner product on $\mathbb{R}^2$, then it should satisfy the equation in part (a). However, if we set $x = (1,0)^T$ and $y = (0,2)^T$ then direct calculation shows that $\|x+y\|_1 = \|x-y\|_1 = 3$, $\|x\|_1 = 1$, and $\|y\|_1 = 2$ which does not satisfy the equation above since $3^2 + 3^2 = 18 \neq 10 = 2(1^2 + 2^2)$.

Question 3. Fundamental Subspaces Revisited .... 20 points

Consider the following matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix}$$

Find an orthogonal (not necessarily orthonormal) basis of each subspace listed below. Here, we assume that $\mathbb{R}^n$ is equipped with the usual dot product.

8 pts (a) $\operatorname{im} A$

12 pts (b) $\ker A$

Answer. The row-reduced echelon form of A is given by $\begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) $v_1 = (1,2,1,2)^T, v_2 = (2,3,3,4)^T$ are a basis of $\operatorname{im} A$. By applying Gram-Schmidt, we have

$$u_1 = v_1 = (1,2,1,2)^T$$
$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = (2,3,3,4)^T - \frac{19}{10}(1,2,1,2)^T = (\frac{1}{10}, -\frac{4}{5}, \frac{11}{10}, \frac{1}{5})^T$$

(b) $v_1 = (1,-2,1,0,0)^T, v_2 = (2,-3,0,1,0)^T, v_3 = (3,-4,0,0,1)^T$ are a basis of $\ker A$. By applying Gram-Schmidt, we have

$$u_1 = v_1 = (1,-2,1,0,0)^T$$
$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = (2,-3,0,1,0)^T - \frac{4}{3}(1,-2,1,0,0)^T = (\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}, 1, 0)^T$$
$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2$$
$$= (3,-4,0,0,1)^T - \frac{11}{6}(1,-2,1,0,0)^T - (\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}, 1, 0)^T = (\frac{1}{2}, 0, -\frac{1}{2}, -1, 1)^T$$

Question 4. QR **Factorization** .... **15 points**

Consider the following matrix.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 6 \end{pmatrix}$$

10 pts (a) Find the QR factorization of A.

5 pts (b) Find the QR factorization of A^T.

Answer.

(a) Let us apply Gram-Schmidt to the column vectors of A.

$$\begin{aligned}
u_1 &= v_1 = (2,1,1)^T \\
u_2 &= v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = (0,1,-1)^T \\
u_3 &= v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2 \\
&= (0,0,6)^T - (2,1,1)^T + 3(0,1,-1)^T = (-2,2,2)^T \\
\therefore Q &= \begin{pmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\
R = Q^T A &= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{6} & 0 & \sqrt{6} \\ 0 & \sqrt{2} & -3\sqrt{2} \\ 0 & 0 & 2\sqrt{3} \end{pmatrix}
\end{aligned}$$

(b) A^T is already upper-triangular with positive diagonal entries, i.e. we may set $Q = I$ and $R = A^T$.

Question 5. Orthonormal Bases .... 15 points

Find an orthonormal basis of $\mathbb{R}^3$ for each inner product.

5 pts (a) $\langle x, y\rangle = x^T \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{pmatrix} y.$

10 pts (b) $\langle x, y\rangle = x^T \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} y.$

Answer.

(a) The given inner product is a weighted dot product. Thus, we may choose $u_1 = (\frac{1}{2}, 0, 0)^T, u_2 = (0, \frac{1}{3}, 0)^T$, and $u_3 = (0, 0, \frac{1}{5})^T$.

(b) Let us apply Gram-Schmidt to $v_1 = (1, 0, 0)^T, v_2 = (0, 0, 1)^T$, and $v_3 = (0, 1, 0)^T$. Then,

$$
\begin{aligned}
u_1 &= v_1 = (1, 0, 0)^T \\
u_2 &= v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = (0, 0, 1)^T \\
u_3 &= v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2 \\
&= (0, 1, 0)^T - \frac{-1}{2}(1, 0, 0)^T - \frac{-1}{2}(0, 0, 1)^T = (\frac{1}{2}, 1, \frac{1}{2})^T
\end{aligned}
$$

Thus after normalization we have

$$\frac{u_1}{\|u_1\|} = (\frac{1}{\sqrt{2}}, 0, 0)^T, \frac{u_2}{\|u_2\|} = (0, 0, \frac{1}{\sqrt{2}})^T, \frac{u_3}{\|u_3\|} = (\frac{1}{2}, 1, \frac{1}{2})^T.$$

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Question 6. Orthogonal Matrices and Matrix Norm .... 20 points

Let $\| \; \| : \mathbb{R}^n \to \mathbb{R}$ be the standard Euclidean norm, i.e.

$$\left\|(a_1, a_2, \ldots, a_n)^T\right\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Recall its associated natural matrix norm

$$\| \; \|_M : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R} : A \mapsto \max\left\{\|Au\| \mid \|u\| = 1\right\}.$$

10 pts (a) Prove the following statement: $Q \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is an orthogonal matrix if and only if $\|Qu\| = 1$ for any unit vector $u \in \mathbb{R}^n$.

5 pts (b) Prove that if $Q \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is orthogonal then $\|Q\|_M = 1$.

5 pts (c) Find a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that $\|A\|_M = 1$ but A is not orthogonal, and explain why your example satisfies the desired conditions. (Here, you may choose n to be any element in $\{1, 2, 3, \ldots\}$ that you want.)

Answer.

(a) If Q is an orthogonal matrix, then $Q^TQ = I$ thus $\|Qu\|^2 = u^TQ^TQu = u^Tu = 1$ for any unit vector $u \in \mathbb{R}^n$. Conversely, if $\|Qu\|^2 = 1$ for any unit vector $u \in \mathbb{R}^n$ then $\|Qx\|^2 = \|x\|^2$ for any $x \in \mathbb{R}^n$ by homogeneity, which means $x^TQ^TQx = x^Tx$. Now we consider the following cases.

- If we set $x = e_i$, then $e_i^TQ^TQe_i = e_i^Te_i = 1$. It means that the diagonal entries of Q^TQ are all 1.
- If we set $x = e_i + e_j$ for $i \neq j$, then we have

$$\begin{aligned} 2 &= (e_i + e_j)^T(e_i + e_j) = (e_i + e_j)^TQ^TQ(e_i + e_j) \\ &= e_iQ^TQe_i + e_iQ^TQe_j + e_jQ^TQe_i + e_jQ^TQe_j = 2 + 2e_iQ^TQe_j \end{aligned}$$

  since $e_iQ^TQe_i = e_jQ^TQe_j = 1$ and $e_iQ^TQe_j = (e_iQ^TQe_j)^T = e_j^TQ^TQe_i$. Thus $e_iQ^TQe_j = 0$ and the (i, j)-entry of Q^TQ is zero when $i \neq j$.

From this it follows that $Q^TQ = I$ as desired.

(b) By part (a), we have $\{\|Qu\| \mid \|u\| = 1\} = \{1\}$. Thus $\|Q\|_M = \max\{1\} = 1$.

(c) Set $n = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then A is not orthogonal as A is not invertible. However, for any $u = (a, b) \in \mathbb{R}^2$ such that $a^2 + b^2 = 1$, we have $\|Au\| = a^2$. Thus, $\{\|Au\| \mid \|u\| = 1\} = \{a^2 \mid a^2 + b^2 = 1\} = [0, 1]$. It follows that $\|A\|_M = \max[0, 1] = 1$.

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