Math 1271 final exam

Question 1. True or False ...................................................... 21 points

For each statement below, determine whether it is true or false and briefly explain why.

3 pts (a) Any continuous function has its derivative.

3 pts (b) A function can have two different horizontal asymptotes.

3 pts (c) If |f(x)| is continuous at x = a, then so is f(x).

3 pts (d) If neither \( \lim_{x \to a} f(x) \) nor \( \lim_{x \to a} g(x) \) exists (as a finite value), then \( \lim_{x \to a} (f(x) + g(x)) \) does not exist (as a finite value).

3 pts (e) If \( f'(c) = 0 \), then \( f(x) \) has a local maximum or minimum at \( x = c \).

3 pts (f) If \( f(x) \) is increasing on \( \mathbb{R} \), then \( f(x)^2 \) is increasing on \( \mathbb{R} \).

3 pts (g) If \( f(x) \) has a discontinuity at 0, then \( \int_{-1}^{1} f(x) dx \) does not exist (as a finite value).

(a) False. Continuity does not imply differentiability.
(b) True. e.g. \( f(x) = \arctan x \).
(c) False. If \( f(x) = 1 \) on \( x \geq 0 \) and \( f(x) = -1 \) on \( x < 0 \), then \( |f(x)| \) is constant but \( f(x) \) is not continuous at 0.
(d) False. e.g. \( f(x) = 1/x, \ g(x) = -1/x, \ a = 0 \)
(e) False. Critical values do not need to be local maximum/minimum. e.g. \( f(x) = x^3, \ c = 0 \)
(f) False. e.g. \( f(x) = -e^{-x} \) It requires both \( f(x) \) and \( f'(x) \) to be positive.
(g) False. For example, if \( f(x) = 1 \) for \( x \neq 0 \) and \( f(0) = 0 \), then \( \int_{-1}^{1} f(x) dx = 2 \).
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Question 2. Graph and Limit .................................................. 23 points

Suppose that the graph of \( y = f(x) \) is given as follows.

For each expression below, calculate its value if it exists or explain why the expression is not well-defined. Please briefly justify your answer.

(a) \( \lim_{x \to 5} f(x) \)  
(b) \( f'(5) \)
(c) \( \lim_{x \to 6} \frac{f(x)}{x - 6} \)
(d) \( \lim_{x \to 2} \frac{x - 2}{f(x)} \)
(e) \( \int_{4}^{6} f(x) \, dx \)
(f) \( \int_{0}^{3} xf(x) \, dx \)

(a) It is 1.2 since the left and right limits are the same.
(b) It does not exist as \( f \) is not continuous at 5.
(c) It is -1.2; by the definition of a derivative (or by l’Hospital’s rule), the answer is the same as \( f'(6) \).
(d) It is 0. The numerator goes to zero and the denominator is bounded, so one may use the squeeze theorem or calculate directly and show that the left and the right limits are both zero.
(e) It is 2.4; it is the area of the right triangle whose legs are 2 and 2.4, respectively.
(f) We have \( \int_{0}^{3} xf(x) \, dx = \int_{0}^{2} 3.5x \, dx + \int_{2}^{3} 6x \, dx = \frac{3.5}{2} x^2 \bigg|_{0}^{2} + 3x^2 \bigg|_{2}^{3} = 22. \)
Question 3. Absolute Maximum/Minimum

For each given function below and given domain of \( x \), find its absolute maximum/minimum values if they exist, or explain why they do not exist. You need to justify your answers.

(a) \( f(x) = x^4 + 5x^3 + 16x^2 + 5, \quad x \in \mathbb{R} \)

(b) \( f(x) = x^2e^x, \quad -4 \leq x \leq 1 \)

(a) \( f'(x) = 4x^3 + 15x^2 + 32x = x(4x^2 + 15x + 32) \) and 0 is the unique root of this since \( 15^2 - 4 \cdot 4 \cdot 32 < 0 \). So 0 is the only critical number and we have \( f(0) = 5 \). Since \( f''(0) = 32 > 0 \), it means that \( f(0) \) is a local minimum, and it should be a global minimum as well. On the other hand, it does not have any global maximum since \( \lim_{x \to \infty} f(x) = \infty \), i.e. the function is not bounded.

(b) \( f'(x) = 2xe^x + x^2e^x \), so \( f'(x) = 0 \) if and only if \( x = 0, -2 \). Here we have four critical numbers: -4, -2, 0, 1. We have \( f(-4) = 16e^{-4}, f(-2) = 4e^{-2}, f(0) = 0, \) and \( f(1) = e \). By direct comparison, we see that \( f(0) \) is the absolute minimum and \( f(1) \) is the absolute maximum.

(Or one may argue as follows: on the other hand, \( f'(x) > 0 \) when \( x \in (-4, -2) \cup (0, 1) \) and \( f'(x) < 0 \) if \( x \in (-2, 0) \), so at least we know that \( f(-4) < f(-2), f(-2) > f(0), \) and \( f(0) < f(1) \). Thus either \( f(-4) \) or \( f(0) \) is the minimum and \( f(-2) \) or \( f(1) \) is the maximum. \( f(0) \) is the minimum since \( f(x) > 0 \) if and only if \( x \neq 0 \). Also \( f(1) \) is the maximum by comparing numbers \( 4e^{-2} \) and \( e \) directly since \( e = 2.718... > 4^{1/3} \).)
Question 4. Solids of Revolution ........................................... 25 points

Please answer the following questions. You need to justify your answers.

(a) Find the volume of the solid obtained by rotating the region $A$ about the $x$-axis.

(b) Find the volume of the solid obtained by rotating the region $A$ about the $y$-axis.

(c) Find the volume of the solid obtained by rotating the region $B$ (i.e. the region circumscribed by $x = 0, y = 0, y = \sqrt{4.8}$, and $y = \sqrt{4.8 \ln x}$) about the $x$-axis.

(d) Explain why the volumes of the solids obtained by rotating the region $C$ and $D$, respectively, about the $y$-axis, should be equal.

\begin{align*}
\text{(a)} \quad & \text{Using the cross-section method, the volume is given by } \int_{0}^{4.3} \pi x^2 \, dx = \frac{\pi}{3} x^3 \bigg|_0^{4.3} = \frac{17.507}{3} \pi. \\
\text{(b)} \quad & \text{Using the cylindrical shell method, the volume is given by } \int_{0}^{4.3} 2\pi x^2 \, dx = \frac{2\pi}{3} x^3 \bigg|_0^{4.3} = \frac{35.014}{3} \pi. \\
\text{(c)} \quad & \text{Using the cylindrical shell method, the volume is given by } \int_{0}^{\sqrt{4.8}} 2\pi ye^{y^2/4.8} \, dy = 4.8\pi e^{y^2/4.8} \bigg|_0^{\sqrt{4.8}} = 4.8\pi(e - 1). \\
\text{(d)} \quad & \text{From the cylindrical shell method, the volumes of two solids are given by } \int_{1}^{e} 2x\pi(\sqrt{4.8 \ln x} - \sqrt{1.2 \ln x}) \, dx = \int_{1}^{e} 2x\pi\sqrt{1.2 \ln x} \, dx \text{ and } \int_{1}^{e} 2x\pi\sqrt{1.2 \ln x} \, dx \ \text{respectively, which are equal. The key point is that } \sqrt{4.8 \ln x} = 2\sqrt{1.2 \ln x}. 
\end{align*}
Question 5. Related Rates and Optimization .......................... 17 points

Please answer the following questions. You need to justify your answers.

5 pts (a) Suppose that two variables $x$ and $y$ satisfy the equation $19x^2 + 32y^2 = C$ for some constant $C$. Find $\frac{dy}{dx}$ at $(x,y) = (10,5)$.

6 pts (b) Find the point $P$ on the line $y = 2x + 5$ which is closest to the origin $O$ and calculate the length of $\overline{OP}$ (the distance between the origin and $P$).

6 pts (c) Each side of a square is increasing at a rate of 8cm/s. At what rate is the area of the square increasing when the area of the square is 25cm$^2$?

(a) We have $38x + 64y \frac{dy}{dx} = 0$, thus $\frac{dy}{dx}_{(x,y)=(10,5)} = -\frac{19x}{32y}_{(x,y)=(10,5)} = -\frac{19}{16}$.

(b) We want to minimize the squared distance $f(x) = x^2 + y^2 = x^2 + (2x + 5)^2 = 5x^2 + 20x + 25$. As $f'(x) = 10x + 20$ is zero only at $x = -2$ and $f''(x) = 10 > 0$, $f(x)$ reaches the global minimum at $x = -2$ and the point is $(-2,1)$. The distance is given by $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$.

(c) The area of a square is $A(x) = x^2$, thus $\frac{dA}{dt} = 2x \frac{dx}{dt}$. When the area is 25, $x = 5$, thus $\frac{dA}{dt} |_{A(x)=25} = 2 \cdot 5 \cdot 8 = 80$. 