Self-Adjoint Operators and Zeros of L-functions

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http://math.umn.edu/~kling202/Research.html
Approaches to RH

Pólya-Hilbert:  
To prove the Riemann Hypothesis, find a self-adjoint operator for which zeros of $\zeta(s)$ are spectral parameters for eigenvalues $\lambda_s = s(s - 1)$. 

Alain Connes' approach:
Form a self-adjoint operator whose eigenvalues $s(s - 1)$ are exactly given by zeros $s$ of $\zeta(s)$ and then prove its self-adjointness via Guinand-Weil explicit formulas.

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The Story

1977  Haas attempted to numerically solve

\[(\Delta - \lambda_s)u = 0\]

on \(\Gamma \setminus \mathcal{H}\) for \(\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\) and \(\lambda_s = s(s - 1)\).
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1981 Hejhal realized Haas had actually solved

$$(\Delta - \lambda_s)u = \delta^\text{afc}_\omega$$

for $\delta^\text{afc}_\omega$ the automorphic Dirac delta at $\omega = e^{2\pi i/3}$. 
Now what? (ColinDeVerdière 1982-83)

1976 Lax & Phillips: For $a > 1$ and

$L^2_a(\Gamma \setminus \mathcal{H}) = \{ f \mid c_P(f(x)) = \int_0^1 f(x + iy) \, dx = 0 \text{ for } y > a \}$,

$\tilde{\Delta}_a$ has purely discrete spectrum

where $\tilde{\Delta}_a$ is the Friedrichs’ extension of $\Delta$
restricted to $L^2_a(\Gamma \setminus \mathcal{H}) \cap C_\infty(\Gamma \setminus \mathcal{H})$ on $L^2(\Gamma \setminus \mathcal{H})$. 

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What happened to the continuous spectrum?

$$f = \sum_{F \text{ cusp}} \langle f, F \rangle F + \frac{\langle f, 1 \rangle \cdot 1}{\langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{(1/2)} \langle f, E_s \rangle E_s ds$$

for $f \in L^2(\Gamma \backslash \mathcal{H})$ decomposed wrt $\Delta$

$$\wedge^a E_s = \left\{ \begin{array}{ll} E_s & y \leq a \\ E_s - (y^s - c_s y^{1-s}) & y > a \end{array} \right.$$ 

(on the fundamental domain) where $c_s = \frac{\xi(2s-1)}{\xi(2s)}$
Refining Observations

\[
(\tilde{\Delta}_a - \lambda_s)u = 0 \iff (\Delta - \lambda_s)u = c \cdot \eta_a \& \eta_a u = 0
\]
for some constant \(c\) and \(\eta_a f = c_P f(ia)\)
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\[\text{proj}_{nc} \delta^{afc}_\omega = \theta \in H^{-3/4-\epsilon} \subseteq H^{-1}\]

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Theorem 0: Bombieri & Garrett (refining observations made by Hejhal and ColinDeVerdière) [2011]

discrete spectrum \(\lambda_s = s(s - 1)\) of \(\tilde{\Delta}_\theta\) (if any!) has spectral parameters \(s\)

\(\subseteq\)

online zeros of \(\zeta(s)L(s, \chi_{-3})\)
On the Inequality

The boundary condition $\eta_a u_s = 0$ gives

$$c_p(u_s) = \frac{(a^s + c_s a^{1-s})y^{1-s}}{1 - 2s} = 0$$

where $c_s = \frac{\xi(2s - 1)}{\xi(2s)}$.

Given $\varepsilon > 0$, for log log $T$ sufficiently large, the minimum spacing between spectral parameters $s$ and $s'$ is at least $(1 - \varepsilon)\pi / \log T$. 
On the Inequality

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Given \( \varepsilon > 0 \), for \( \log \log T \) sufficiently large, the minimum spacing between spectral parameters \( s \) and \( s' \) is at least \( (1 - \varepsilon) \pi / \log T \).

Exotic eigenfunction expansions and regular spacing of \( \zeta \) on the edge of the critical strip is in conflict with Montgomery’s Pair Correlation:

Montgomery PCC:

\[
\sum_{0 < \gamma, \gamma' \leq T} 1 \sim \frac{T}{2\pi} \log T \int_{0}^{\beta} \left( 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 \right) du
\]

for \( 1/2 + i\gamma \) and \( 1/2 + i\gamma' \) zeros of \( \zeta \).
More concisely...

**Theorem 1:** Bombieri & Garrett [2013]
Assuming Montgomery’s pair correlation conjecture at most 94% of the nontrivial zeros of $\zeta(s)$ can appear as spectral parameters $s$ for $\lambda_s$. 
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Assuming Montgomery’s pair correlation conjecture at most 94% of the nontrivial zeros of $\zeta(s)$ can appear as spectral parameters $s$ for $\lambda_s$.

*all or nothing?*
Negative results?

Check ‘simplest’ possible case ✓

Clarifying the details alone took 30 years. Even then a conclusion is reached only by some serious operator-theory, Montgomery pair correlation and regular edge behavior of $\zeta(s)$. 
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Investigate other possible more complicated boundary-value problems
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What is the potential scope of such results?
A Noncompact Period

Let $[\tilde{k} : k] = 2$. 
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$E_s$ an Eisenstein series on $G = \text{Res}_k^{\tilde{k}}(GL_2(\tilde{k}))$ \cup $f$ a cuspform on $H = GL_2(k)$
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L-function:
\[ \int_{Z_\mathbb{A}H_k \backslash H_\mathbb{A}} f(g) \cdot \text{Res}_{H}^{G} E_s(g) \, dg \]
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↑ by standard unwinding
and local multiplicity-one results
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Let

$E = \text{subset of } L^2(Z_\mathbb{A} G_k \backslash G_\mathbb{A}) \text{ gen by Eisenstein series with trivial grossencharacter,}$

$$\int_{(1/2)} A_s \cdot E_s = F \in E \text{ and } \theta : F \to \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} f \cdot F$$

so that $\theta E_s$ is an L-function.
Results

**Key Lemma:**

\[ \theta \left( \int_{(1/2)} A_s \cdot E_s \ ds \right) = \int_{(1/2)} A_s \cdot \theta E_s \ ds \]
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\& \quad 2^{nd} \text{ moment bound } \Rightarrow \theta \in H^{-1}(Z_A G_k \setminus G_A).
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2\textsuperscript{nd} moment bound $\Rightarrow \theta \in H^{-1}(Z_A G_k \setminus G_A)$.

**1\textsuperscript{st} Main Result:**

Let $S = \tilde{\Delta}_\theta$,

discrete spectrum of $S$ (if any) has spectral parameters $s \subseteq$ online zeros of $\theta E_s$. 
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1^{st} Main Result:
Let \( S = \tilde{\Delta}_\theta \),
discrete spectrum of \( S \) (if any) has spectral parameters \( s \subseteq \) online zeros of \( \theta E_s \).

2^{nd} Main Result:
Exotic eigenfunction expansion and regular edge behavior of \( \zeta_{\tilde{\kappa}}(s) \) and pair correlation yields a strict inequality.
Thanks!