Spectral Theory

Kim Klinger-Logan

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ABSTRACT: The following is a collection of notes (one of many) that compiled in preparation for my Oral Exam which related to spectral theory for automorphic forms. None of the information is new and it is rephrased from Rudin's *Functional Analysis*, Evans' *Theory of PDE*, Paul Garrett's online notes and various other sources.

Background on Spectra

To begin, here is some basic terminology related to spectral theory.

For a continuous linear operator $T \in \text{End}X$, the **λ-eigenspace** of T is

$$X_\lambda := \{x \in X \mid Tx = \lambda x\}.$$

If $X_\lambda \neq 0$ then λ is an **eigenvalue**. The **resolvent** of an operator T is $(T - \lambda)^{-1}$. The resolvent set

$$\rho(T) := \{\lambda \in \mathbb{C} \mid \lambda \text{ is a regular value of } T\}.$$

The value λ is said to be a **regular value** if $(T - \lambda)^{-1}$ exists, is a bounded linear operator and is defined on a dense subset of the range of T.

The spectrum of is the collection of λ such that there is no continuous linear resolvent (inverse). In oter works $\sigma(T) = \mathbb{C} - \rho(T)$.

spectrum $\sigma(T) := \{\lambda \mid T - \lambda \text{ does not have a continuous linear inverse}\}$

discrete spectrum $\sigma_{disc}(T) := \{\lambda \mid T - \lambda \text{ is not } \textit{injective} \text{ i.e. not invertible}\}$

continuous spectrum $\sigma_{cont}(T) := \{\lambda \mid T - \lambda \text{ is not } \textit{surjective but} \text{ does have a dense image}\}$

residual spectrum $\sigma_{res}(T) := \{\lambda \in \sigma(T) - (\sigma_{disc}(T) \cup \sigma_{cont}(T))\}$

Bounded Operators

Usually we want to talk about bounded operators on Hilbert spaces. Recall that a **Hilbert space** is a vector space with an inner product space that is complete with respect to the distance function induced by the metric. We can sometimes deduce nice properties for operators on **Banach spaces** (a complete normed metric space) when the operator satisfies more specific conditions but most of the time we will stick with Hilbert spaces.

A linear map $T : X \to Y$ is **bounded** if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $|x|_X < \delta$ then $|Tx|_Y < \epsilon$. For a linear map of Hilbert spaces, the following are equivalent:

(i) continuous

(ii) continuous at 0

(iii) bounded.

The **adjoint** of a map $T : X \to Y$ is the map $T^* : Y \to X$ so that $\langle Tx, y\rangle_Y = \langle x, T^*y\rangle_X$.

Any *continuous* linear map T have a *unique* adjoint T^* and $T^{**} = T$.

For a self-adjoint maps, all eigenvalues are *real*.

If $T \in \text{End}X$ and $T^*T = TT^*$ then T is **normal**. For a normal operator there is no residual spectrum–it is all discrete or continuous.

Any *normal* operator has $\sigma_{res}(T) = \emptyset$.

A normal operator is **unitary** if $T^*T = TT^* = I$.

Compact Operators

All compact operators are bounded operators. Thus everything that has previously been said still applies. There are many ways to define a **compact operator** $T : X \to Y$:

- maps the unit ball in X to a precompact (has compact closure) set in Y
- maps bounded subsequences in X to sequences in Y with convergent subsequences.

Let $\{T_n\}$ be a sequence of compact operators on a normed linear space X. Suppose that $T_n \to T$ (in the space of bounded operators). Then T is also compact.

For compact operators we do not need to restrict the domain to Hilbert spaces in order to have some spectral theorem. We have a spectral theorem for compact operators on Banach spaces. One method of proof involves the following two Lemmas:

REISZ'S LEMMA: Let X be a Banach space and $Y \subset X$ a proper *closed* subspace. For all $\epsilon > 0$ there is an $x \in X$ with $||x|| = 1$ and

$$1 \geq d(x, Y) \geq 1 - \epsilon$$

(We can think of this as the analogue to orthogonality in Hilbert spaces but for Banach spaces where there is no notion of orthogonality.)

proof: Let $x_1 \notin Y$ and set $R := \inf_{y\in Y} |y - x_1|$. Note that $R > 0$.

For $\epsilon > 0$ let $y_1 \in Y$ so that $|x_1 - y_1| < R + \epsilon$.

Set $x := \dfrac{x_1 - y_1}{y_1 - x_1}$ and note that $||x|| = 1$ and

$$\inf_{y\in Y} |y - x| = \inf_{y\in Y} \left| y - \frac{x_1 - y_1}{y_1 - x_1} \right| = \inf_{y\in Y} \left| y + \frac{y_1}{y_1 - x_1} - \frac{x_1}{y_1 - x_1} \right| = \frac{\inf_{y\in Y} |y - x_1|}{|y_1 - x_1|} = \frac{R}{R + \epsilon}$$

We can make this quotient arbitrarily close to 1 by taking smaller ϵ.

We can think of this as the analogue to orthogonality in Hilbert spaces but for Banach spaces where there is no notion of orthogonality.

LEMMA 2: If T is compact then $\text{Im}(T - I)$ is closed.

We can also use the Fredholm Alternative to prove the first part of the Spectral Theorem for Compact Operators.

> FREDHOLM ALTERNATIVE: Let $T : X \to X$ be a compact operator on Banach spaces and $\lambda \neq 0$, either
>
> (A) $T - \lambda$ is a bijection *or*
>
> (B) $\text{Im}(T - \lambda)$ is closed and $\dim(X/\text{Im}(T - \lambda)) = \dim(\ker(T - \lambda))$.

We can also use Reisz' Lemma to prove the Fredholm Alternative.

SPECTRAL THEOREM FOR **Compact Operators** ON *Banach Spaces*:
Let T be a compact on an infinite-dimensional Banach space then

(1) the nonzero spectrum is *discrete* (for both T and T^*)
(2) $\lambda_i \to 0$ as $i \to \infty$ (and this is the only accumulation point)
(3) the spectrum is countable
(4) the number of eigenvalues outside a disk $|\lambda| \leq r$ is finite for $r \geq 0$

proof:

(1) *This follows nicely from the Fredholm Alternative:*

Suppose that $\lambda \neq 0$ is in $\sigma(T)$ but is not an eigenvalue for T. Then $T - \lambda$ is injective. Thus $\dim(\ker(T - \lambda)) = 0$ and by the Fredholm Alternative, $\dim(X/\text{Im}(T - \lambda)) = 0$. Thus $T - \lambda$ is surjective and so λ must be in the resolvent.

We can also prove this result using Reisz' Lemma:

The idea of the proof is by contradiction. Also WLOG, assume $\lambda = 1$.

Assume $\lambda \in \sigma(T)$ is *not* an eigenvalue. Thus $T - \lambda = T - I$ is injective but not surjective. Since T is compact, by LEMMA 2, $\text{Im}(T - I)$ is a closed proper subspace of X. Call this $Y_1 := \text{Im}(T - I)$.

Since $T - I$ is *injective*, $Y_2 := (T - I)Y_1$ is a closed proper subspace of Y_1.

Create a strictly decreasing sequence of $Y_n := \text{Im}(T - I)^n$ so

$$Y_1 \supset Y_2 \supset \cdots \supset Y_n \supset \cdots \supset Y_m \supset \ldots$$

Now apply REISZ'S LEMMA for $Y := Y_{n+1}$ and $\epsilon = 1/2$: Choose $y_n \in Y_n$ so that $||y_n|| = 1$ and $d(y_n, Y_{n+1}) > 1/2$.

Also note that by compactness of T, $\{Ty_n\}$ must contain a norm convergent subsequence.

But for $n < m$,

$$||Ty_n - Ty_m|| = ||(T-I)y_n + y_n - (T-I)y_m - y_m||$$

and $(T-I)y_n - (T-I)y_m - y_m \in Y_{n+1}$ and thus $||Ty_n - Ty_m|| > 1/2$ which is our contradiction. Thus λ must be an eigenvalue.

(2) This will also proceed by contradiction using Reisz' Lemma.

Assume for the sake of contradiction that there are infinitely many distinct eigenvalues λ_n outside a ball centered at the origin of radius ϵ and each λ_n has associated eigenvector x_n.

Define $Y_n := \text{span}\{x_1, x_2, \ldots, x_n\}$.

This gives a strictly increasing sequence

$$Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots \subset Y_m \subset \ldots$$

Now apply Reisz' Lemma for $Y = Y_{n-1}$ and $\epsilon = 1/2$: Choose $y_n \in Y_n$ so that $||y_n|| = 1$ and $d(y_n, Y_{n-1}) > 1/2$.

Again by compactness of T, $\{Ty_n\}$ must contain a norm convergent subsequence.

But for $n < m$,

$$||Ty_n - Ty_m|| = ||(T-\lambda_n)y_n + \lambda_n y_n - (T-\lambda_m)y_m + \lambda_m y_m||$$

where $(T-\lambda_n)y_n + \lambda_n y_n - (T-\lambda_m)y_m \in Y_{m-1}$ since $y_m = \sum_{i=1}^{m} c_i x_i$ where $\sum_{i=1}^{m-1} c_i x_i \in Y_{m-1}$ and $(T-\lambda_m)x_m = 0$. Thus $||Ty_n - Ty_m|| > \epsilon/2$. This is a contradiction. Thus there are only finitely many eigenvalues outside a ball centered at 0. This gives us that zero is the only accumulation point.

(3) From (2),

$$\{\lambda_n\} = \bigcup_n \{|\lambda| > 1/n\}$$

where each of these sets is finite. Thus $\sigma(T)$ is countable. □

Let $T : X \to Y$ be a compact operator where X is a Banach space and Y is a Hilbert space, Then T is the limit (in operator norm) of a sequence of finite-dimensional operators.

More can be said about the operators on Hilbert spaces–especially when they are self-adjoint. For self-adjoint operators (or even symmetric) on Hilbert spaces **all of the eigenvalues are real**:

Let T be a self-adjoint operator on a Hilbert space with eigenvalue λ so that $Tx = \lambda x$. Then

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle$$
$$\overline{\langle Tx, x\rangle} = \overline{\langle \lambda x, x\rangle} = \overline{\lambda}\langle x, x\rangle$$

Thus $\lambda = \overline{\lambda}$ and so $\lambda \in \mathbb{R}$.

Spectral Theorem for **Self-Adjoint Compact Operators** on a *Hilbert Space*:

Let T be a compact self-adjoint operator on a Hilbert space H, then

- H has an orthonormal basis v_i of eigenvectors of T and $H =$ the completion of $\bigoplus_\lambda H_\lambda$.
- $\lambda_i \to 0$ as $i \to \infty$ (and this is the only accumulation point)
- every eigenspace H_λ is finite-dimensional
- (Ralleigh-Ritz) either $\pm|T|_{op}$ is an eigenvalue.

Recall that $|T|_{op} := \inf\{c \geq 0 \mid |Tx| \leq c|x| \text{ for all } x \in X\}$.

Example: *Finite-dimensional*

Let $T : X \to Y$ be a continuous linear mapping between normed spaces. If $\text{Ran}(T)$ has finite dimension then T is called a *finite-dimensional operator*.

Finite dimensional operators are compact.

For a bounded set $B \subseteq X$, $\overline{T(B)}$ is closed and bounded in the finite-dimensional subspaces $\text{Ran}(T) \subseteq Y$. Heine-Borel says that $\overline{T(B)}$ is compact in $\text{Ran}(T)$.

Example: *Hilbert-Schmidt*

Hilbert Schmidt Theorem: Let X be a locally compact space endowed with a positive Borel measure and assume that $L^2(X)$ is a separable Hilbert space. Let $K \in L^2(X \times X)$ (i.e. $\displaystyle\int_X \int_X |K(x,y)|^2 \, dxdy < \infty$). Then the operator

$$(Tf)(x) = \int_X K(x,y) f(y) \, dy$$

is a compact operator on $L^2(X)$.

The basic idea for this proof is to write T as a limit of finite dimensional operators. This operator T is called the **Hilbert-Schmidt operator** and K is called the *Hilbert-Schmidt kernel*.

Spectral Theorem applied to Elliptic Operators

Consider the eigenvalue problem for the Laplacian with Dirichlet Boundary Conditions on a smooth bounded domain U:

$$\begin{cases} (\Delta - \lambda)u = 0 & \text{on } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{1}$$

First we must establish the existence of an inverse to Δ. We can do this by showing the existence of a solution operator for the following system

$$\begin{cases} \Delta u = f & \text{on } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{2}$$

where f is a continuous linear function on H_0^1 so $f \in H^{-1}$. The associated bilinear form for Δ is

$$B[u, v] = \int_U \nabla u \cdot \nabla v \; dx$$

(the inner product in H_0^1). Recall that *weak solutions* of (2) are $u \in H_0^1$ such that for all $v \in H_0^1$,

$$B[u, v] = \langle f, v \rangle.$$

For a general elliptic operator we would using Energy Estimates, we see that the hypothesis for Lax-Milgram are met and thus by LAX-MILGRAM we can say that there is a unique weak solution to (2). However, since Δ is symmetric this just follows directly from the REISZ REPRESENTATION THEOREM by making $(u, v); = B(u, v)$ the new inner product on H.

> LAX-MILGRAM: Assume that $B : H \times H \to \mathbb{R}$ is a bilinear form for which there are constants $\alpha, \beta > 0$ so that
>
> $$|B[u, v]| \leq \alpha ||u|| \cdot ||v||$$
>
> for $u, v \in H$ and $|| \cdot ||$ the Hilbert space norm and
>
> $$\beta ||u||^2 \leq B[u, u]$$
>
> for $u \in H$. Let $f : H \to \mathbb{R}$ be a bounded linear functional on H.
>
> Then there is a *unique* u so that $B[u, v] = \langle f, v \rangle$ for all $v \in H$.

The proof of LAX-MILGRAM essentially boils down to showing that f can be represented by u in $B[\cdot, \cdot]$ using the REISZ REPRESENTATION THEOREM. This is very simple in the case of Δ since

> REISZ REPRESENTATION THEOREM: H^* can be cannonically identified with H; more precisely for all $u^* \in H^*$ there is a unique $u \in H$ so that $\langle u^*, v \rangle = (u, v)$ for all $v \in H$ (where $\langle \cdot, \cdot \rangle$ is the pairing of H^* and H and $(\cdot, \cdot)$ is the H-inner product).
>
> The linear map $u^* \mapsto u$ is a linear isomorphism.

Thus there is an inverse operator to Δ, call it $K : H^{-1} \to H_0^1$ that maps $f \mapsto u$ (which is bounded). Since $L^2 \subset H^{-1}$ and H_0^1 compactly embeds into L^2 (Rellich compactness), for $f \in L^2$ we can rename the solution operator K to be the composition of a compact map with a continuous map

$$K : L^2 \to H_0^1 \to L^2$$

so we have that K is compact. Furthermore, K is a self-adjoint operator on Hilbert spaces. Thus we are able to apply the Spectral Theorem for Self-Adjoint Compact Operators on Hilbert Spaces to K.

We can then say of Δ that its eigenfunctions are equal to those of K and its eigenvalues $\lambda_n = 1/\mu_n$ for μ_n and eigenvalue of K. Thus for Δ (and the same can be shown more generally for any symmetric elliptic operator using roughly the same method):

Spectral Theorem for Δ:

- all eigenvalues are real
- If we repeat each eigenvalue according to its finite multiplicity, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ and $\lambda_k \to \infty$ as $k \to \infty$.
- There exists an orthonormal basis $\{u_k\}$ of $L^2(U)$ where $U_k \in H_0^1$ is an eigenfunction corresponding to λ_k.

It is important to note that the theorem above only applies to *symmetric* elliptic operators. When an elliptoc operator is not symmetric, there is no guarantee that all of the eigenvalues will be real.

Unbounded Operators

A linear map $T : D \to Y$ is **unbounded** if it is not bounded and may not be defined on all of X for $D \subset X$. A notion related to self-adjointness exists in this case. Not much can be said about unbounded operators that are not densely-defined or at least closed. A **closed** operator can be naturally defined by having a closed graph in $X \oplus Y$. Explicitly we can also say that every sequence $\{x_n\}$ in D converging to $x \in X$ such that $Tx_n \to y \in Y$ as $n \to \infty$ we have $x \in D$ and $Tx = y$. Closed operators are more general than bounded operators – they are not necessarily continuous. Note that not every closed operator is densely defined. However, every densely-defined operator has a closed adjoint.

An operator T is **symmetric** when $T \subset T^*$. Note that all self-adjoint operators are symmetric but not vice versa.

- If T is *symmetric* then $T \subset T^{**} \subset T^*$.
- If T is *closed and symmetric* then $T = T^{**} \subset T^*$.
- If T is *self-sdjoint* then $T = T^{**} = T^*$.
- If T is *essentially self-sdjoint* then $T \subset T^{**} = T^*$.

The class of self-adjoint operators is especially important in mathematical physics. Every self-adjoint operator is densely-defined, closed and symmetric. The converse holds for bounded operators but fails in general. An operator is *self-adjoint* if it is densely-defined, closed symmetric *and both* $T + i$ *and* $T - i$ *are surjective.*

The spectral theorem applies to self-adjoint operators and moreover, to normal operators, but not to densely defined, closed operators in general, since in this case the spectrum can be empty:

> Spectral Theorem for Unbounded Operators: Any multiplication operator is a (densely-defined) self-adjoint operator. Any self-adjoint operator is unitarily equivalent to a multiplication operator.
>
> Let H be a separable Hilbert space and let $(D(T), T) \in DD^*(H)$ be a self-adjoint operator on H. Then there exists a finite measure space (X, μ), a measurable real-valued function $g : X \to R$, and a unitary map $U : L^2(X, \mu) \to H$ such that $U^{-1}TU = M_g$, the operator of multiplicative by g, i.e., such that
>
> $$U^{-1}(D(T)) = D(M_g) = \{\varphi \in L^2(X, \mu) \mid g\varphi \in L^2(X, \mu)\},$$
>
> and
>
> $$U(g\varphi) = T(U(\varphi))$$
>
> for $\varphi \in D(M_g)$.
>
> Moreover, we have $\sigma(T) = \sigma(Mg) = \text{supp}(g * (\mu))$, which is the essential range of g. The only difference with the bounded case is therefore that the spectrum, and the function g, are not necessarily bounded anymore.

The only difference with the bounded case is therefore that the spectrum, and the function g, are not necessarily bounded anymore.

> *Eigenvalues of symmetric operators are real.*

Compact Resolvent

For an unbounded operator T on a Hilbert space, if there is a $\lambda \in \rho(T)$ so that $R_\lambda = (T-\lambda)^{-1}$ is a compact operator then we say that T has **compact resolvent**.

> If an unbounded operator T on a Hilbert space has *compact resolvent* then the spectrum of T is discrete.

proof: By the Spectral Theorem for Compact Operators, all nonzero numbers in the spectrum of a compact operator R_{λ_0} are eigenvalues of finite multiplicity which have no nonzero accumulation point. Also it can be shown that

> LEMMA: For $\lambda_0 \in \rho(T)$ and $\lambda \neq \lambda_0$,
>
> λ is an eigenvalue of $T \iff (\lambda - \lambda_0)^{-1}$ is an eigenvalue for $R_{\lambda_0}(T)$ (with the same multiplicities).

By this lemma, T has purely discrete spectrum.

> If T is also *self-adjoint* the spectrum is real (in $\mathbb{R}$) and there exists and orthonormal basis of eigenvectors. (The eigenvalues have no accumulation point though.)

Let $R_\lambda = (T-\lambda)^{-1}$ be the resolvent for $\lambda \in \mathbb{C}$ when this inverse exists as a linear operator defined at least on a dense subset of V.

THEOREM: Let T be self-adjoint and densely defined. For $\lambda \in \mathbb{C} - \mathbb{R}$ the operator R_λ is everywhere defined on V, and the operator norm is estimated by

$$||R_\lambda|| \leq \frac{1}{|\text{Im}\lambda|}$$

For T positive, $\lambda \notin [0,\infty)$, R_λ is everywhere defined on V and the operator norm is estimated by

$$||R_\lambda|| \leq \begin{cases} \frac{1}{|\text{Im}\lambda|} & \text{Re}(\lambda) \leq 0 \\ \frac{1}{|\lambda|} & \text{Re}(\lambda) \geq 0 \end{cases}$$

THEOREM: (Hlibert) For points λ, μ off the real line, or, for T positive, for λ, μ off $[0,\infty)$

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu$$

For the operator-norm topology, $\lambda \to R_\lambda$ is holomorphic at such points.

Something stronger can be said if the operator is defined on a dense subset of X and is **positive** (i.e. $\langle Tv, v\rangle \geq 0$ for all $v \in D$).

> FRIEDRICHS EXTENSION: A densely defined, positive, symmetric operator has a self-adjoint extension.

Friedrichs Extension

A semibounded symmetric operator is one which is satisfies $\langle Sv, v\rangle \geq c \cdot \langle v, v\rangle$ or $\langle Sv, v\rangle \leq c \cdot \langle v, v\rangle$ for some constant $c > 0$. The operator $1 - \Delta$ is an canonical example of such an operator since $\langle \Delta f, g\rangle \geq 0$. We can construct the Friedrichs' extension of a densely-defined, symmetric operator S as follows:

Without loss of generality, consider a densely-defined, symmetric operator S with domain D_S with

$$\langle Sv, v\rangle \geq \langle v, v\rangle$$

for all $v \in D_S$. (Note that any semibounded operator can be exhibited this way by multiplying by a constant and adding or subtracting a constant.)

Define the inner product $\langle , \rangle_1$ on D_S by

$$\langle v, w\rangle_1 := \langle Sv, w\rangle$$

for $v, w \in D_S$ and let V^1 be the completion of D_S with respect to the metric induced by $\langle , \rangle_1$. Since $\langle v, v\rangle_1 \geq \langle v, v\rangle$, the inclusion map $D_S \hookrightarrow V$ extends to a continuous map $V^1 \hookrightarrow V$. Furthermore, since D_S is dense in V, we have that V^1 is also dense in V.

For $w \in V$, the functional $v \mapsto \langle v, w\rangle$ is a continuous linear functional on V^1 with norm

$$\sup_{|v|_1 \leq 1} |\langle v, w\rangle| \leq \sup_{|v|_1 \leq 1} |v| \cdot |w| \leq \sup_{|v|_1 \leq 1} |v|_1 \cdot |w| = |w|.$$

By the **Riesz-Frechet Theorem** on V^1, there is a $w' \in V^1$ so that

$$\langle v, w'\rangle_1 = \langle v, w\rangle$$

for all $v \in V^1$ and $w \in V$ with norm bounded by the norm of $v \mapsto \langle v, w\rangle$; explicitly, $|w'|_1 \leq |w|$. The map $A : V \to V^1$ defined by $w \mapsto w'$ is linear. The inverse of A will be a self-adjoint extension $\widetilde{S}$ of S. This is the *Friedrichs extension*. We now show that $\widetilde{S}$ is in fact self-adjoint and an extension of S.

First note that since $|Aw|_1 = |w'|_1 \leq |w|$ from above, the operator norm is $\sup_{|w|\leq 1} |Aw|_1 \leq 1$ and so A is continuous.

Also observe that for $w' \in D_S$ and all $v \in V^1$,

$$\langle v, w'\rangle_1 = \langle v, Sw'\rangle = \langle v, A(Sw')\rangle_1$$

so $A(Sw') = w'$ for each $w' \in D_S$. Hence $AV \subset V^1$ contains the domain D_S of S.

We also see that A is injective since $\ker A = 0$: since V^1 is dense in V, if $0 = \langle v, Aw\rangle_1 = \langle v, w\rangle$ for all $v \in V^1$ then $w = 0$. Thus the inverse $\widetilde{S}$ of A is defined on $D_{\widetilde{S}} = AV \subset V^1$. Hence $\widetilde{S}$ is injective and is surjective for $D_{\widetilde{S}} \to V$.

Now to show that $\widetilde{S}$ is an extension of S, it remains to show that $A(\widetilde{S}w) = A(Sw)$ for $w \in D_S$. For $v, w \in D_S \subset D_{\widetilde{S}}$,

$$\langle v, \widetilde{S}\rangle = \langle v, A(\widetilde{S}w)\rangle_1 = \langle v, w\rangle_1 = \langle Sv, w\rangle = \langle v, Sw\rangle.$$

Since D_s is dense in V we have that $\widetilde{S}w = Sw$.

We also must show that $\widetilde{S}$ is symmetric. First note that A is symmetric: for $w' = Aw \in AV$ since $\langle v, Aw\rangle_1 = \langle v, w\rangle$ we have $\langle v, w'\rangle_1 = \langle v, \widetilde{S}w'\rangle$ and

$$\langle Av, w\rangle = \langle Av, \widetilde{S}Aw\rangle = \langle Av, Aw\rangle_1$$

which is symmetric in v and w. Thus since $D_{\widetilde{S}} = AV$ and

$$\langle \widetilde{S}Av, Aw\rangle = \langle v, Aw\rangle = \langle Av, w\rangle = \langle Av, Aw\rangle_1 = \langle Av, \widetilde{S}Aw\rangle$$

and so $\widetilde{S}$ is symmetric.

Furthermore, this extension $\widetilde{S}$ remains semibounded i.e. $\langle \widetilde{S}v, v\rangle \geq \langle v, v\rangle$ for all $v = Aw \in D_{\widetilde{S}} = AV$ since

$$\langle \widetilde{S}v, v\rangle = \langle \widetilde{S}Aw, v\rangle = \langle w, v\rangle = \langle Aw, v\rangle_1 = \langle v, v\rangle_1 \geq \langle v, v\rangle.$$

It remains to show that $\widetilde{S}$ is self-adjoint. Note that any proper extension $T \supset \widetilde{S}$ is not injective since $\widetilde{S}$ surjects to V. So if S^* were a proper extension of $\widetilde{S}$ there would be $v \in D_{\widetilde{S}}$ so that for all $w \in D_{\widetilde{S}}$,

$$0 = \langle \widetilde{S}^* v, w\rangle = \langle v, \widetilde{S}w\rangle = \langle v, \widetilde{S}w\rangle.$$

Since $\widetilde{S}$ surjects to V, there is a $w \in D_{\widetilde{S}}$ such that $\widetilde{S}w = v$. Hence $v = 0$ and $\widetilde{S}^*$ cannot be a proper extension of $\widetilde{S}$. Thus $\widetilde{S}$ is self-adjoint.

This construction serves as a proof for the following theorem of Friedrichs (?, ?).

Theorem 1. *A positive, densely-defined, symmetric operator S with domain D_S has a positive self-adjoint extension with the same lower bound.*

This extension has useful properties of particular interest to our project. An alternative characterization of the extension makes this more clear.

Assume that V has a $\mathbb{C}$-conjugate-linear complex conjugation $v \to v^c$ with the properties: $(v^c)^c = v$ and $\langle v^c, w^c\rangle = \overline{\langle v, w\rangle}$. Further let S commute with conjugation so that $(Sv)^c = S(v^c)$. Let V^{-1} be the dual of V^1 so that $V^1 \subset V \subset V^{-1}$.

Note that given this small specification, there is an alternate characterization of the Friedrichs extension. To specify it, define a continuous, complex-linear map $S^\# : V^1 \to V^{-1}$ by

$$(S^\# v)(w) = \langle v, w^c\rangle_1$$

for $v, w \in V^1$.

Theorem 2. *Let $X = \{v \in V^1 \mid S^\# v \in V\}$. Then the Friedrichs extension of S is $\widetilde{S} = S^\#|_X$ with domain $D_{\widetilde{S}} = X$.*

Proof. Let $T = S^{\#}|_X$. Let $: V \to V^1$ be the inverse of $\widetilde{S}$ defined by $\langle Av, w\rangle_1 = \langle v, w\rangle$ for all $w \in V^1$ and $v \in V$ from the Riesz-Fischer Theorem. Then

$$\langle TAv, w\rangle = \langle Av, w\rangle_1 = \langle v, w\rangle$$

for $v \in V$ and $w \in V^1$. Also,

$$\langle ATv, w\rangle_1 = \langle Tv, w\rangle = \langle v, w\rangle_1$$

for $v \in X$ and $w \in V^1$. This $T = A^{-1} = \widetilde{S}$. □

Extensions of Restrictions

Using the latter characterization of the Friedrichs extension we will now explain how the construction of the extension works with the case of restricted operators. Assume that S and the related terms are defined as above. Let $\Theta \subset D_S$ be a S-stable subspace. Let the orthogonal complement to Θ in V be

$$\ker \Theta = \{v \in V \mid \langle v, \theta\rangle = 0 \text{ for all } \theta \in \Theta\}.$$

For our purposes, given an operator S as above, we will want to define $T = S|_{D_S \cap \ker \Theta}$ so that $D_T = D_S \cap \ker \Theta$.

Note that for $v \in D_T$ and $\theta \in \Theta$,

$$\langle Tv, \theta\rangle = \langle Sv, \theta\rangle = \langle v, S\theta\rangle \in \{\langle v, \theta'\rangle \mid \theta' \in \Theta\} = \{0\}$$

and so $T(D_T) \subset \ker \Theta$. Furthermore since T is a restriction of S, symmetry and $\langle Tv, v\rangle \geq \langle v, v\rangle$ are inherited from S.

In contrast with these inherited properties, it is nontrivial to give a simple condition to ensure that D_T is dense in $\ker \Theta$ and that the V^1-closure of D_T is $V^1 \cap \ker \Theta$. This delicacy is demonstrated in Lax and Phillips (?, ?). For cut-off above height $a > 1$, an argument using the geometry of the fundamental domain for Γ shows that $\Theta \cap V^1$ is dense in V.[1] For this reason we will *assume* $D_T = D_S \cap \ker \Theta$ is V-dense in $\ker \Theta$ and V^1-dense in $V^1 \cap \ker \Theta =: W^1$.

Let W^{-1} be the dual of W so we have $W^1 \subset \ker \Theta \subset W^{-1}$. Define $S^{\#} : V^1 \to V^{-1}$ by

$$(S^{\#}v)(w) := \langle v, w^c\rangle_1,$$

for $v, w \in V^1$.

Theorem 3. *Let $\Theta^{\#}$ be the V^{-1}-completion of Θ. The Friedrichs extension $\widetilde{T}$ of T has domain $D_{\widetilde{T}} = \{v \in W^1 \mid S^{\#}v \in V + \Theta^{\#}\}$ and is characterized by*

$$\widetilde{T}v = w \quad \iff \quad S^{\#}v \in w + \Theta^{\#}$$

for $v \in D_{\widetilde{T}}$ and $w \in ker\Theta$.

[1]For $a < 1$, there are serious complications so we do not address this case.

Proof. Define $T^{\#} : W^1 \to W^{-1}$ by

$$(T^{\#}v)(w) := \langle v, w^c \rangle_1$$

we can then define the the domain of the Friedrichs extension $\widetilde{T}$ as

$$D_{\widetilde{T}} = \{w \in W^1 \mid T^{\#}w \in \ker \Theta\}$$

so that $\widetilde{T} = T^{\#}|_{D_{\widetilde{T}}}$. With the inclusion $j : W^1 \to V^1$ for all $x, y \in W^1$

$$(T^{\#}x)(y) = \langle jx, (jy)^c \rangle_1 = (S^{\#}jx)(jy) = \big((j^* \circ S^{\#} \circ j)x\big)(y)$$

and so $T^{\#} = j^* \circ S^{\#} \circ j$ and

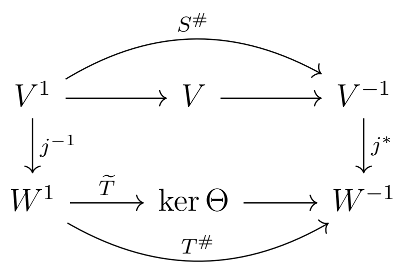
$$D_{\widetilde{T}} = \{w \in W^1 \mid j^*(S^{\#}(jw)) = 0\}.$$

The orthogonal compliment $\ker \Theta$ to Θ in V is a closed subspace of V^1 and the dual W^{-1} of W^1 is

$$W^{-1} = (V^1 \cap \ker \Theta)^* \cong V^{-1}/\Theta^{\#}.$$

$\square$

The Friedrichs extension makes the following diagram commute



Spectral Theory of Automorphic Forms

There are several examples of automorphic forms on $\Gamma\backslash\mathfrak{H}$:

(a) Holomorphic modular forms

(b) Maass forms

(c) Constant functions

Representation theory helps us with the question:

> QUESTION 1: Why are precisely these the types of automorphic forms on $\Gamma\backslash\mathfrak{H}$ and no others?

> QUESTION 2: Where do differential operators come from and why do they work?

The answer to QUESTION 2 also comes from representation theory–namely Lie theory and we will address it later.

However QUESTION 1 is also a question related to spectral theory in that Maass forms are eigenfunctions for the *Maass operators* on $C^\infty(\mathfrak{H})$

$$R_k = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2} = (z-\overline{z})\frac{\partial}{\partial z} + \frac{k}{2}$$

$$L_k = -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{k}{2} = (z-\overline{z})\frac{\partial}{\partial \overline{z}} - \frac{k}{2}.$$

Note that it is easily verified that

$$\Delta_k = -L_{k+2}R_k - \frac{k}{2}\left(1+\frac{k}{2}\right) = -R_{k-2}L_k + \frac{k}{2}\left(1-\frac{k}{2}\right).$$

The Laplacian Δ_k is a symmetric operator on $L^2(\mathfrak{H})$ with domain $C_o^\infty(\mathfrak{H})$. Recall that the SPECTRAL THEOREM only applies to self-adjoint operators; however, even for symmetric operators we can conclude that if f is an L^2 eigenfunction so that $\Delta_k f = \lambda f$ then λ is real and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Also since $\mathfrak{H}$ has infinite volume, the Laplacian has too many eigenfunctions to be of real interest. A more interesting and tangible situations is to consider the decomposition when $\Gamma\backslash\mathfrak{H}$ is compact or finite volume.

$\Gamma\backslash\mathfrak{H}$ compact

There are two versions of the spectral problem that are closely related:

> SPECTRAL PROBLEM (VERSION 1): Determine the spectrum of the unbounded symmetric operator Δ_k on $L^2(\Gamma\backslash\mathfrak{H}, \chi, k)$.

Here $L^2(\Gamma\backslash\mathfrak{H},\chi,k)$ is the Hilbert space completion of the space of smooth functions on $\mathfrak{H}$ such that

$$\chi(\gamma)f(z) = \left(\frac{c\bar{z}+d}{|cz+d|}\right)^k f\left(\frac{az+b}{cz+d}\right)$$

for $\gamma \in \Gamma$, $C^\infty(\Gamma\backslash\mathfrak{H},\chi,k)$.

> Spectral Problem (version 2): Determine the decomposition of the Hilbert space $L^2(\Gamma\backslash G,\chi)$ into irreducible subspaces.

Where $L^2(\Gamma\backslash G,\chi)$ is the square-integrable functions satisfying

$$f(\gamma g u) = \chi(\gamma)f(g)$$

for $\gamma \in \Gamma$, $u \in Z^+$ and $g \in G$ that are square integrable with respect to the Haar measure on G/Z^+.

Relating the Spectral Problems

To relate these two versions of the spectral problems, we need to describe how these two spaces $L^2(\Gamma\backslash\mathfrak{H},\chi,k)$ and $L^2(\Gamma\backslash G,\chi)$ related to one another. Define the *right regular representation*

$$\rho : G \to \mathrm{End}\left(L^2(\Gamma\backslash G,\chi)\right)$$

meaning that $(\rho(g)f)(x) = f(xg)$ for $g, x \in G$. This is a unitary representation.

It is common to restrict the action of ρ on to the maximal compact K of G. In doing so, we have that

$$L^2(\Gamma\backslash G,\chi) = \bigoplus_{k\in\mathbb{Z}} L^2(\Gamma\backslash G,\chi,k)$$

where $L^2(\Gamma\backslash G,\chi,k)$ is the subspace consisting of functions such that $\rho(\kappa_\theta)f = e^{ik\theta}f$ (or $f(g\kappa_\theta) = e^{ik\theta}f(g)$). There is a Hilbert space isomorphism

$$\sigma_k : L^2(\Gamma\backslash\mathfrak{H},\chi,k) \to L^2(\Gamma\backslash G,\chi,k)$$

given by $(\sigma_k f)(g) = (f|_k g)(i)$ for $g \in G$.

In order to understand the relationship between these Hilbert Spaces, we need to define operators on $L^2(\Gamma\backslash G,\chi)$ to play the role of Δ_k, R_k and L_k. We define the following differential operators on G

$$R = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\right),$$

$$L = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

and the *Laplace-Beltrami operator*

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x\partial\theta}.$$

The direct relationship between these two versions of the spectral problem can best be seen in the following theorem:

Theorem 4. *Let Γ be a discontinuous subgroup of G such that $-I \in \Gamma$ and $\Gamma\backslash G/K$ is compact. Let χ be a character of G. Let $\chi(-1) = (-1)^{\epsilon}$ where $\epsilon = 0$ or 1.*

(a) The space $H = L^2(\Gamma\backslash G, \chi)$ decomposes into a Hilbert space direct sum of irreducible representations.

The spaces $L^2(\Gamma\backslash\mathfrak{H}, \chi, k)$ each decompose into a Hilbert space direct sum of eigenspaces for Δ_k.

(b) Let H_k be any subspace of H that is invariant under the action of G. Then H_k is also invariant under the action of Δ.

Conversely, let $\lambda \in \mathbb{R}$ and let H_λ be the λ-eigenspace of Δ on H, then H_λ is G-invariant.

(c) If H_k is an irreducible subspace of H then Δ acts as a scalar on H_k.

The eigenvector λ of Δ on H_k depends only on the isomorphism class of H_k. If λ is such an eigenvalue than λ is a real number. Either $\lambda \geq 0$ is $\epsilon = 0$ and $\lambda \geq 1/4$ if $\epsilon = 1$, otherwise $\lambda = \frac{k}{2}(1 - \frac{k}{2})$ where $1 \leq k \in \mathbb{Z}$ and $k \equiv \epsilon \bmod 2$.

(d) There is only one finite-dimensional representation of G that can occur in H–the trivial representation:

If $\chi = 1$ then the constant function spans a one-dimensional irreducible subspace of H on which space the eigenvalue λ of Δ equals 0.

All other irreducible constituents of H are infinite dimensional.

(e) Assume λ is not of the form $\frac{k}{2}(1 - \frac{k}{2})$ where $1 \leq k \in \mathbb{Z}$ and $\equiv \epsilon \bmod 2$:

There exists a unique irreducible representation $\mathcal{P}(\lambda, \epsilon)$ of G depending only on λ (and not Γ) such that if H_k is an infinite-dimensional irreducible subrepresentation of H with eigenvalue λ then $H_k \cong \mathcal{P}(\lambda, \epsilon)$.

Let $k \equiv \epsilon \bmod 2$ be an integer. The multiplicity of the representation $\mathcal{P}(\lambda, \epsilon)$ is equal to the multiplicity of the eigenvalue in $L^2(\Gamma\backslash\mathfrak{H}, \chi, k)$.

(f) Assume that λ is of the form $\frac{k}{2}(1 - \frac{k}{2})$ where $1 \leq k \in \mathbb{Z}$ and $k \equiv \epsilon \bmod 2$.

There exists two irreducible representations, $\mathcal{D}^+(k)$ and $\mathcal{D}^-(k)$ of G depending only on k such that if H_k is an infinite-dimensional irreducible representation of H with λ then either $H_k \cong \mathcal{D}^+(k)$ or $H_K \cong \mathcal{D}^+(k)$. The representations $\mathcal{D}^\pm(k)$ have the same multiplicity in H; this multiplicity equals the dimension of the space of holomorphic modular forms of weight k that satisfy

$$f(\gamma z) = \chi(\gamma)(cz + d)^k f(z)$$

for $\gamma \in \Gamma$.

Part (a) above unifies these two versions. However understanding the link between the spectral and representation theory can best be seen in the proof of this result.

'Proof' of (a):
Let χ be a unitary character of Γ and $H = L^2(\Gamma\backslash G, \chi)$ Let $\phi \in C_c^\infty(G)$ and define

$$\rho(\phi) : H \to H$$

by $\rho(\phi)f(g) = \int_G f(gh)\phi(h)\ dh$. Thus we have $\rho(\phi)f = \int_G \phi(h)\rho(h)f\ dh$ where ρ is a right-regular representation.

the following are some properties of such representations:

Proposition 5. *Let $\phi \in C_c^\infty(G)$.*

(a) The operator $\rho(\phi)$ is a Hilbert-Schmidt operator. In particular, the operator is compact.
If $f \in\ L^2(\Gamma\backslash G, \chi)$ then $\rho(\phi)f \in C^\infty(\Gamma\backslash G, \chi)$.

(b) If $\phi(g) = \phi(g^{-1})$ then the operator $\rho(\phi)$ is self-adjoint.
More generally, if $\pi : G \to End(H)$ is a unitary representation of G on a Hilbert space H and if $\phi(g) = \overline{\phi(g^{-1})}$ then $\pi(\phi)$ is self-adjoint.

(c) For $\kappa_\theta \in K$, if $\phi(\kappa_\theta g) = e^{-ik\theta}\phi(g)$ then $\rho(\phi)$ maps the Hilbert space $L^2(\Gamma\backslash G, \chi)$ into $C^\infty(\Gamma\backslash G, \chi, k)$.

Define $\pi(\phi) \in \mathrm{End}(V)$ by

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v\ dg.$$

We can use the above properties to establish the following lemma.

Lemma 6. *Let $\pi : G \to End(H)$ be a unitary representation of G on a Hilbert space H and let $0 \neq f \in H$. Let $\epsilon > 0$ be given. Then there exists $\phi \in C_c^\infty(G)$ such that $\pi(\phi)$ is self-adjoint and $|\pi(\phi)f - f| < \epsilon$.*
In particular if $\epsilon < |f|$ this implies that $\pi(\phi)f \neq 0$.
Moreover if $\pi(\kappa_\theta) = e^{ik\theta}f$ for all $\kappa_\theta \in K$, then we may choose ϕ so that $\phi(\kappa_\theta g) = \theta(g\kappa_\theta) = e^{ik\theta}\phi(g)$.

If ρ is *unitary* and it is *reducible* (it has a a proper nonzero closed subspace V) then the orthogonal complement U of V is also a nonzero invariant closed subspace and $H = U \oplus V$. On the other hand, if it is not unitary, it is possible for there to be a closed invariant subspace V that is not complimented.

From this lemma we have

Proposition 7. *Let H be a nonzero closed subspace of $L^2(\Gamma\backslash G, \chi)$ which is closed under the action of G. Then H has decomposition as a Hilbert space direct sum*

$$H = \bigoplus_k H_k$$

where $\rho(\kappa_\theta)f = e^{2\pi ik\theta}f$ for $f \in H_k$.
Let k be such that $H_k \neq 0$ then Δ has a nonzero eigenvector in $H_k \cap C^\infty(\Gamma\backslash G, \chi)$.

To prove this result we use the fact that $\rho(\phi)$ is a compact operator from Proposition 5 and the Spectral Theorem for compact operators.
From this we get the first part of (a):

Theorem 8. *The space $L^2(\Gamma\backslash G, \chi)$ decomposes into Hilbert space direct sum of subspaces that are invariant and irreducible under the right regular representation ρ.*

Similarly we get

Theorem 9. *Let ξ be a character of $C_c^\infty(K\backslash G/K,\sigma)$ and let $H(\xi)$ be the space of $f \in L^2(\Gamma\backslash G,\chi,k)$ that satisfys $\pi(\phi)f = \xi(\phi)f$ for all $\phi \in C_c^\infty K\backslash G/K,\sigma)$. The space $H(\xi)$ is a finite dimensional subspace of $C^\infty(\Gamma\backslash G,\chi,k)$.*
If ξ and η are two distinct characters of $C_c^\infty(K\backslash G/K,\sigma)$ then $H(\xi)$ and $H(\eta)$ are orthogonal subspaces. Furthermore,

$$L^2(\Gamma\backslash G,\chi,k) = \bigoplus_\xi H(\xi)$$

where $H(\xi) \neq 0$.

Using this theorem and the fact that because Δ commutes with the operators in $C_c^\infty(K\backslash G/K,\sigma)$ the $H(\xi)$ are Δ-invariant and so Δ induces a self-adjoint operator on each of the finite dimenstional vector spaces $H(\xi)$ so each of these decomposes into a direct sum of Δ-eigenspaces.

Corollary 10. *The space $L^2(\Gamma\backslash\mathfrak{H},\chi,k)$ decomposes into a Hilbert space direct sum of eigenspaces of Δ_k.*

Both these results that make up part (a) follow from Proposition 5.

Spectral Decomposition for $\Gamma = SL_2(\mathbb{Z})$

In this case, $\Gamma\backslash\mathfrak{H}$ is non-compact. Thus there is no reason to suspect that the spectrum is purely discrete.We will now exhibit the computation of this spectral decomposition for $L^2(\Gamma\backslash\mathfrak{H})$ with respect to $\Delta = y^2\left(\partial_x^2 + \partial_y^2\right)$.

Let N be the upper-triangular unipotent matrices in $G = SL_2(\mathbb{R})$, A the diagonal matrices, A^+ the diagonal matrices with positive diagonal entries, $P = NA$ the parabolic subgroup of upper-triangular matrices, $P^+ = NA^+$, $\Gamma = SL_2(\mathbb{Z})$ and $\Gamma_\infty = P^+ \cap \Gamma = N \cap \Gamma$. For simplicity, normalize the total measure of K to 1 rather than 2π.

Pseudo-Eisenstein Series

Pseudo-Eisenstein series are solutions to the *adjunction problem*: given $\varphi \in C_c^\infty(N\backslash G)$, we want to find $\Psi_\varphi \in C_c^\infty$ such that

$$\langle c_P f, \varphi\rangle_{N\backslash G} = \langle f, \Psi_\varphi\rangle_{\Gamma\backslash G}$$

for f on $\Gamma\backslash G$ and $\langle f, F\rangle_{\Gamma\backslash G} = \int_{\Gamma\backslash G} f \cdot F$.

We can compute the canonical expression for Ψ_φ from this desired equality using the left N-invariance of φ and the left Γ-invariance of f as follows: (Note that $P\cap\Gamma$ differs from $N\cap\Gamma$ only by $\pm 1_2$ which act trivially on $\mathfrak{H} \cong G/K$.)

$$\langle c_P f, \varphi\rangle_{N\backslash\mathfrak{H}} = \int_{N\backslash\mathfrak{H}} c_P f(z)\varphi(\mathrm{Im}(z))\,\frac{dx\,dy}{y^2} = \int_{N\backslash\mathfrak{H}} \left(\int_{N\cap\Gamma\backslash N} f(nz)\,dn\right)\varphi(\mathrm{Im}(z))\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma_\infty\backslash\mathfrak{H}} f(z)\varphi(\mathrm{Im}(z))\,\frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f(\gamma z)\varphi(\mathrm{Im}(\gamma z))\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} f(z) \left(\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \varphi(\mathrm{Im}(\gamma z)) \right) \frac{dx\ dy}{y^2} = \langle f, \Psi_\varphi \rangle_{\Gamma\backslash\mathfrak{H}}$$

Thus we define the *pseudo-Eisenstein series* as $\Psi_\varphi(z) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \varphi(\mathrm{Im}(\gamma z))$. Note that the pseudo-Eisenstein series is absolutely and uniformly convergent for $z \in C$ where C is a compact subset of G. Furthermore, $\Psi_\varphi \in C_c^\infty(\Gamma\backslash G)$.

Now, it is a corollary of the above characterization of psuedo-Eisenstein series that the square integrable cuspforms are the orthogonal complement of the (closed) subspace of $L^2(\Gamma\backslash\mathfrak{H})$ spanned by pseudo-Eisenstein series with $\varphi \in C_0^\infty(N\backslash\mathfrak{H}) \cong C_0^\infty(0,\infty)$. Thus we have

$$L^2(\Gamma\backslash\mathfrak{H}) = L^2_{\text{cusp}}(\Gamma\backslash\mathfrak{H}) \oplus L^2_{\text{p-Eis}}(\Gamma\backslash\mathfrak{H}).$$

Decomposition of Pseudo-Eisenstein Series

We further decompose $L^2(\Gamma\backslash\mathfrak{H})$ by examining the pseudo-Eisenstein series Ψ_φ. The spectral decomposition of the data φ induces a spectral decomposition for Ψ_φ. Identifying $N\backslash\mathfrak{H} \cong N\backslash G/K \cong A^+$, Mellin inversion gives

$$\varphi(\mathrm{Im}\ z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)(\mathrm{Im}\ z)^s\ ds$$

for any real σ. This decomposition of φ is achieved as follows.

Replacing ξ by $\xi/2\pi$ in Fourier inversion we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-it\xi}\ dt \right) e^{i\xi x}\ d\xi.$$

Fourier transforms on $\mathbb{R}$ put into multiplicative coordinates are *Mellin transforms*:
For $\varphi \in C_c^\infty(0,\infty)$, take $f(x) = \varphi(e^x)$. Let $y = e^x$ and $r = e^t$ (the exponential in the implied inner integral) and rewrite Fourier inversion as

$$f(x) = \varphi(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \varphi(r) r^{-i\xi} \frac{dr}{r} \right) y^{i\xi}\ d\xi$$

since $dt = \frac{dr}{r}$. Note that this integral converges as a C^∞-function-valued function.

The Fourier transform (inner integral) in these coordinates is Mellin transform. For compactly supported φ, the integral definition extends to all $s \in \mathbb{C}$ as $\mathcal{M}\varphi(s) = \int_0^\infty \varphi(r) r^{-s} \frac{dr}{r}$. Mellin inversion is

$$\varphi(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}\varphi(i\xi) y^{i\xi}\ d\xi.$$

With ξ the imaginary part of a complex variable s, we can rewrite this as a complex path integral

$$\varphi(y) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \mathcal{M}\varphi(s) y^s\ ds$$

since $d\xi = -i\ ds$. For $f \in C_c^\infty(\mathbb{R})$, $\hat{f}(\xi)$ converges nicely for all complex values of ξ so it extends to an entire function in ξ of rapid decay on horizontal lines (Payley-Wiener Theorem). This extension property applies to φ allowing us to move the contour as above.

Thus the pseudo-Eisenstein series is

$$\Psi_\varphi(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\mathrm{Im}(\gamma z)) = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{M}\varphi(s) \cdot (\mathrm{Im}(\gamma z))^s\ ds.$$

It would be natural to take $\sigma = 0$ however, at $\sigma = 0$ the double integral would not be absolutely convergent and the two integrals cannot be interchanged. For $\sigma > 1$ the double integral is absolutely convergent and (using Fubini) we have,

$$\Psi_\varphi(z) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{M}\varphi(s) \cdot E_s(z)\ ds$$

for $E_s(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}(\gamma z))^s$ the Eisenstein series.

Eisenstein Series

We have a decomposition of Ψ_φ in terms of φ. We now want to rewrite this piece of the decomposition so as to refer only to Ψ_φ not φ. Note that E_s has meromorphic continuation on the entire complex plane. Thus we can move the line of integration to the left and choose $\sigma = 1/2$ to achieve

$$\Psi_\varphi = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \mathcal{M}\varphi(s) E_s\ ds + \sum_{s_0} \mathrm{res}_{s=s_0}(\mathcal{M}\varphi(s) E_s).$$

As with the pseudo-Eisenstein series, the Eisenstein series E_s fits into an adjunction relation

$$\langle E_s, f \rangle_{\Gamma \backslash \mathfrak{H}} = \langle y^s, c_P f \rangle_{\Gamma_\infty \backslash \mathfrak{H}}$$

for f on $\Gamma \backslash \mathfrak{H}$. Notice that

$$\begin{aligned}
\langle y^s, c_P f \rangle_{A^+} &= \int_{N \backslash \mathfrak{H}} c_P f(z) \cdot y^s\ \frac{dx\ dy}{y^2} = \int_{N \backslash \mathfrak{H}} \left(\int_{\Gamma_\infty \backslash N} f(nz)\ dn \right) \cdot y^s\ \frac{dx\ dy}{y^2} \\
&= \int_{\Gamma_\infty \backslash \mathfrak{H}} f(z) \cdot y^s\ \frac{dx\ dy}{y^2} = \int_{P \cap \Gamma \backslash \mathfrak{H}} f(z) \cdot y^s\ \frac{dx\ dy}{y^2} \\
&= \int_{\Gamma \backslash \mathfrak{H}} \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} f(\gamma z) \cdot \mathrm{Im}(\gamma z)^s\ \frac{dx\ dy}{y^2} \\
&= \int_{\Gamma \backslash \mathfrak{H}} f(z) \cdot \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} \mathrm{Im}(\gamma z)^s\ \frac{dx\ dy}{y^2} = \langle E_s, f \rangle_{\Gamma \backslash \mathfrak{H}}.
\end{aligned}$$

Thus

$$\langle E_s, f \rangle_{\Gamma \backslash \mathfrak{H}} = \int_0^\infty c_P f(iy) y^s \frac{dy}{y^2} = \int_0^\infty c_P f(iy) y^{-(1-s)}\ \frac{dy}{y} = \mathcal{M}(c_P f)(1-s).$$

On the other hand, since $c_P E_s = y^s + c_s y^{1-s}$ for $c_s = \dfrac{\xi(2s-1)}{\xi(2s)}$, we have

$$\langle E_s, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} = \langle c_P E_s, \varphi\rangle_{\Gamma_\infty\backslash\mathfrak{H}} = \langle y^s + c_s y^{1-s}, \varphi\rangle_{\Gamma_\infty\backslash\mathfrak{H}} = \int_0^\infty (y^s + c_s y^{1-s})\cdot\varphi(y)\ \frac{dy}{y^2}$$

$$= \int_0^\infty (y^{-(1-s)} + c_s y^{-s})\cdot\varphi(y)\ \frac{dy}{y} = \mathcal{M}\varphi(1-s) + c_s\mathcal{M}\varphi(s).$$

So we have the identity,

$$\mathcal{M}(c_P\Psi_\varphi)(s) = \langle E_{1-s}, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} = \mathcal{M}\varphi(s) + c_{1-s}\mathcal{M}\varphi(1-s).$$

Using this and returning to our equation for Ψ_φ above, we get

$$\Psi_\varphi - \sum_{s_0} \text{res}_{s=s_0}(\mathcal{M}\varphi(s)E_s) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s\ ds$$

$$= \frac{1}{2\pi i}\int_{1/2-i0}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s + \mathcal{M}\varphi(1-s)E_{1-s}\ ds$$

$$= \frac{1}{2\pi i}\int_{1/2-i0}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s + c_{1-s}\mathcal{M}\varphi(1-s)E_s\ ds$$

$$= \frac{1}{2\pi i}\int_{1/2-i0}^{1/2+i\infty} \mathcal{M}c_p\Psi_\varphi(s)E_s\ ds$$

from the functional equation $E_{1-s} = c_{1-s}E_s$ and our identity above

$$= \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \langle\Psi_\varphi, E_s\rangle_{\Gamma\backslash\mathfrak{H}}\cdot E_s\ ds.$$

Residue

Finally, let us examine the residue. For $\Gamma = SL_2(\mathbb{Z})$, the only pole of E_s in the half plane $\text{Re}(s) \geq 1/2$ is at $s_0 = 1$. This pole is simple and the residue is a constant function. Thus we can compute the residue as follows:

$$\sum_{s_0} \text{res}_{s=s_0}(\mathcal{M}\varphi(s)E_s) = \mathcal{M}\varphi(1)\cdot\text{res}_{s=1}E_s$$

where

$$\mathcal{M}\varphi(1) = \int_0^\infty \varphi(y)y^{-1}\frac{dy}{y} = \int_0^\infty \varphi(y)\frac{dy}{y^2} = \int_{N\backslash\mathfrak{H}} \varphi(\text{Im}z)\frac{dx\ dy}{y^2}$$

$$= \int_{N\backslash\mathfrak{H}}\int_{\Gamma_\infty\backslash N} \varphi(\text{Im}(nz))\ dn\ \frac{dx\ dy}{y^2} = \int_{N\backslash\mathfrak{H}} \varphi(\text{Im}(nz))\left(\int_{\Gamma_\infty\backslash N} 1\ dn\right)\frac{dx\ dy}{y^2}$$

$$= \int_{\Gamma_\infty\backslash\mathfrak{H}} \varphi(\text{Im}z)\ \frac{dx\ dy}{y^2}$$

since the volume of $\Gamma_\infty \backslash N$ is 1 and φ is left N-invariant

$$= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\mathrm{Im} z)\ \frac{dx\ dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \Psi_\varphi(z)\ \frac{dx\ dy}{y^2} = \langle \Psi_\varphi, 1\rangle_{\Gamma\backslash\mathfrak{H}}.$$

This we have that

$$L^2(\Gamma\backslash\mathfrak{H}) = L^2_{\mathrm{cusp}}(\Gamma\backslash\mathfrak{H}) \oplus \mathbb{C} \oplus L^2_{\mathrm{Eis}}(\Gamma\backslash\mathfrak{H}).$$