Now that we are much more comfortable calculating limits, we are ready to begin calculating derivatives. *You may want to be aware that this worksheet contains two of the most important concepts of the semester. So it may be beneficial to re-read this handout regularly until the concepts are solidified.

**Derivatives and Rates of Change**

Recall the 2.1 worksheet. Here, we were trying to calculate the instantaneous rate of change of a falling object. This instantaneous rate of change is what we call the **derivative**.

How would you calculate the rate of change of a function \( f(x) \) between the points \( x = a \) and \( x = b \)?

Sketch a picture/graph that describes what is happening in the previous formula.

If \( f \) were a linear function, this value would calculate the **slope** of that line.

Since \( f \) may not be a linear function, we say that this value is calculating the **slope of the secant line** through \((a, f(a))\) and \((b, f(b))\) of \( f \).

However, what we want to find is the **slope of the tangent line** at one particular point \((a, f(a))\). This will give us the **instantaneous** rate of change.

| Definition | Tangent line to the curve \( y = f(x) \) at the point \((a, f(a))\) is the line through \((a, f(a))\) with slope \[ m = \lim_{{x \to a}} \frac{{f(x) - f(a)}}{{x - a}} \] provided that this limit exists. |

Explain why \( m \) models that instantaneous rate of change of \( f \) at \((a, f(a))\). (You may want to use your graph/picture.)
Use the previous definition to find the equation of the tangent line to \( f(x) = x^2 \) at \((2, 4)\).

Now, setting \( h = x - a \) in the previous definition, rewrite the formula for the slope of the tangent line in terms of \( x \) and \( h \).

We call this result the derivative.

**Definition**

The **derivative of a function \( f \) at a number \( a \)**, denoted \( f'(a) \), is denoted

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

if this limit exists.

Because this is just a rewriting of our previous definition of the slope of the tangent line,

the derivative of a function \( f \) at a number \( a \), \( f'(a) \), is the slope of the line tangent to \( f \) at the point \((a, f(a))\).

In other words,

the derivative \( f'(a) \) is also the instantaneous rate of change of \( y = f(x) \) with respect to \( x \) at \( x = a \).
Examples

1. Find an equation for the tangent line to the curve $f(x) = x^2 - 3$ at $x = 3$.
   In order to do this, we must...
   (a) Find the derivative of $f(x) = x^2 - 3$, $f'(x)$, using the definition of the derivative.
   
   (b) Evaluate $f'(3)$.

   (c) Use $f'(3)$ and the point $(3, f(3))$ to find an equation for the tangent line to the curve $f(x) = x^2 - 3$ at $x = 3$.

2. Find an equation for the tangent line to the curve $f(x) = \sqrt{x}$ at $x = 4$. 
3. If a rock is thrown upward on Mars with velocity 10 m/s, its height (in meters) after $t$ seconds is given by $h(t) = 10t - 1.86t^2$.

Recall that the derivative $h'(a)$ is also the instantaneous rate of change of $y = h(x)$ with respect to $x$ at $x = a$. This means it is the instantaneous velocity of $h(x)$ with respect to $x$ at $x = a$.

(a) Find the velocity of the rock when $t = a$

(b) Find the velocity of the rock after one second.

(c) When did the rock hit the ground?

(d) With what velocity will the rock hit the ground?