Important Note: The ideas that we will be discussing on this worksheet are very complex. It will take reading through sentences multiple times in order for their meaning to register. I suggest you do this. The proofs have breaks after each sentence for you to absorb their content (as well as for you to make notes that help you better understand).

Recall our more precise definition of the limit of a function:

Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \) itself at \( a \) itself.

Then we say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and we write

\[
\lim_{x \to a} f(x) = L
\]

if every number \( \epsilon > 0 \) there is a corresponding number \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \epsilon \).

We have already seen that estimating limits using graphs and tables can be problematic. Hence, it would help if we were able to derive some tools to help us compute limits when they appear to be more complicated.

**Limit Laws**

As responsible investigators, we will attempt to establish each of these limit laws. But, don’t worry, we are going to walk through the proofs of a few of the Laws of Limits together.

**Sum Law**

The first Law of Limits is the **Sum Law**. The Sum Law basically states that the limit of the sum of two functions is the sum of the limits.

\[
\text{The Sum Law}
\]

\[
\text{If } \lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M \text{ both exist then}
\]

\[
\lim_{x \to a} [f(x) + g(x)] = L + M.
\]

However, before we can walk through the proof of this law, let’s establish what is called the **Triangle Inequality**.
The Triangle Inequality

If $a$ and $b$ are any real numbers, then

$$|a + b| \leq |a| + |b|.$$ 

proof of the Triangle Inequality: Assume that $a$ and $b$ are real numbers.

What can be said about the relationship between $|a + b|$ and $|a| + |b|$ if $a$ and $b$ are both positive?

Similarly, what can be said about the relationship between $|a + b|$ and $|a| + |b|$ if both $a$ and $b$ are both negative?

Notice that since either $a = |a|$ or $a = -|a|$, $-|a| \leq a \leq |a|$.

Similarly, since either $b = |b|$ or $b = -|b|$, $-|b| \leq b \leq |b|$.

What do you get when you add these inequalities?

Apply the fact that $|x| \leq c$ if and only if $-c \leq x \leq c$ to your result.

What have we established?

Now we are ready to start discussing the proof of the Sum Law.

But before we dive right into the proof, we want to make some preliminary notes.
Preliminary Notes:
In order to prove this law, what we want to establish:

\[
given \text{ that } \lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M \text{ both exist,}
we want to be able to show \lim_{x \to a} [f(x) + g(x)] = L + M.
\]

According to our precise definition of a limit, getting \(\lim_{x \to a} [f(x) + g(x)] = L + M\) means that for any \(\epsilon > 0\),
we must find a \(\delta > 0\) such that if \(0 < |x - a| < \delta\) then \(_______________________________ < \epsilon\).

Using the Triangle Inequality, we can write (*)

\[
|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq ______________________________
\]

This means, if we can make \(|f(x) - L| + |g(x) - M|\) less than \(\epsilon\), \(|f(x) + g(x) - (L + M)| < \epsilon\).

We will make \(|f(x) + g(x) - (L + M)|\) less than \(\epsilon\) by making each of the terms \(|f(x) - L|\) and \(|g(x) - M|\)
less than \(\epsilon/2\).

Now we are ready for the proof...

Proof of the Sum Law:
Let \(\epsilon > 0\).
Assume \(_______________________________\) and \(_______________________________\) both exist.

Since \(\epsilon/2 > 0\) and \(\lim_{x \to a} f(x) = L\), (by the definition of a limit)
there exists a number \(\delta_1 > 0\) such that if \(0 < |x - a| < \delta_1\) then \(|f(x) - L| < ______\).

Similarly, since \(\lim_{x \to a} g(x) = M\),
there exists a number \(\delta_2 > 0\) such that if \(0 < |x - a| < \delta_2\) then \(|g(x) - M| < ______\).

Define \(\delta = \min\{\delta_1, \delta_2\}\), the smaller of the two numbers \(\delta_1\) and \(\delta_2\).
Notice that if \(0 < |x - a| < \delta\) then \(0 < |x - a| < ______\) and \(0 < |x - a| < ______\)
and so \(|f(x) - L| < ______\) and \(|g(x) - M| < ______\).

Therefore, by (*) (that we worked out before),
\[
|f(x) + g(x) - (L + M)| \leq ______________________________ < \epsilon/2 + \epsilon/2 = \epsilon.
\]

Ta-da! We did it!
Why did that prove the Sum Law?

From the Sum Law, we directly get the Difference Law.

The Difference Law

If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) both exist then

\[
\lim_{x \to a} [f(x) - g(x)] = L - M.
\]

Why does this result directly follow from the Sum Law?

It seems to me that you have only learned something if you can turn around and apply it, so now, it’s your turn!

The Product Law

The Product Law basically states that if you are taking the limit of the product of two functions then it is equal to the product of the limits of those two functions.

The Product Law

If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) both exist then

\[
\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M.
\]

The proof of this law is very similar to that of the Sum Law, but things get a little bit messier.

Fill in the following blanks appropriately.
Preliminary Notes:
In order to prove this law, what we want to establish:

\[ \text{given that } \lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M \text{ both exist,} \]

\[ \text{According to our precise definition of a limit, getting } \lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M \text{ means that} \]

\[ \text{We can write (*)} \]

\[ |f(x) \cdot g(x) - (L \cdot M)| \]

\[ \leq \]

\[ 0 \]

\[ \leq |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M| \]

This means, if we can make \(|g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M| \) less than \(\epsilon\), then we will have

\[ \text{We will make } |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M| \text{ less than } \epsilon \text{ by making:} \]

\[ |g(x)| \cdot |f(x) - L| \text{ less than } \epsilon \text{ and } |L| \cdot |g(x) - M| \text{ less than } \epsilon. \]

This means we will make, \(|f(x) - L| \) less than

\[ \frac{\epsilon}{2(1 + |M|)} \]

and \(|g(x) - M| \) less than

\[ \frac{\epsilon}{2(1 + |L|)}. \]
PROOF OF THE PRODUCT LAW:
Let \( \epsilon > 0 \).
Assume \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) both exist.

Since \( \frac{\epsilon}{2(1+|M|)} > 0 \) and \( \lim_{x \to a} f(x) = L \), (by the definition of a limit)

there exists a number \( \delta_1 > 0 \) such that if \( 0 < |x - a| < \delta_1 \) then

\[
|f(x) - L| < \text{______________________________}.
\]

Similarly, since \( \frac{\epsilon}{2(1+|L|)} > 0 \) and \( \lim_{x \to a} g(x) = M \),

there exists a number \( \delta_2 > 0 \) such that if \( 0 < |x - a| < \delta_2 \) then

\[
|g(x) - M| < \text{______________________________}.
\]

Also, let \( |g(x) - M| < 1 \) when \( 0 < |x - a| < \delta_3 \)

Define \( \delta = \text{______________________________} \).

Notice that if \( 0 < |x - a| < \delta \) then \( 0 < |x - a| < \text{___________} \).

\( 0 < |x - a| < \text{___________} \) and \( 0 < |x - a| < \text{___________} \)

and so \( |f(x) - L| < \text{______________________________} \) and

\[
|g(x) - M| < \text{______________________________}.
\]

Therefore, by (*) (that we worked out before),

\[
|f(x) \cdot g(x) - (L \cdot M)|
\]

\[
\leq \text{______________________________}
\]

\[
\leq \text{______________________________} \leq \text{___________}.
\]

You did it! You have now prove the Product Law for Limits.
We noticed previously that from the Sum Law, it is easy to establish the difference law. We now encounter a similar phenomena with the Product Law and the following laws:

If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) both exist then

- **The Constant Multiple Law**
  \[
  \lim_{x \to a} [cg(x)] = cM.
  \]

- **The Quotient Law**
  \[
  \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}.
  \]

- **The Power Law** (where \( n \) is a positive integer)
  \[
  \lim_{x \to a} [f(x)]^n = L^n.
  \]

Explain how to use the Product Law to prove the Constant Multiple Law.

Explain how to use the Product Law to prove the Quotient Law.

Explain how to use the Product Law to prove the Power Law.
Practice Exercises

Given that
\[
\lim_{x \to -1} f(x) = 5 \quad \lim_{x \to -1} g(x) = -3 \quad \lim_{x \to -1} h(x) = 13.5 \quad \lim_{x \to -1} k(x) = 0
\]

find the limit that exists. If the limit does not exist, explain why.

a. \( \lim_{x \to -1} [f(x) + 3g(x)] \)

b. \( \lim_{x \to -1} [g(x)]^3 \)

c. \( \lim_{x \to -1} \frac{3g(x)}{k(x)} \)

d. \( \lim_{x \to -1} \frac{f(x)g(x)}{h(x)} \)

e. \( \lim_{x \to -1} [h(x) - 2f(x)] \)

f. \( \lim_{x \to -1} [f(x) \sqrt[3]{g(x)}] \)