NAME__________________________

Derivatives and Shapes of Curves

In Section 4.2 we discussed how to find the extreme values of a function using the derivative. These results say,

In Chapter 2, we discussed how to find intervals of where a function is increasing and decreasing using the first derivative and intervals where a function is concave up and concave down using the second derivative.

These results say,

<table>
<thead>
<tr>
<th>INCREASING/DECREASING TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) If $f'(x) &gt; 0$ on an interval, then __________________________.</td>
</tr>
<tr>
<td>(2) If $f'(x) &lt; 0$ on an interval, then __________________________.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CONCAVITY TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) If $f''(x) &gt; 0$ on an interval, then __________________________.</td>
</tr>
<tr>
<td>(2) If $f''(x) &lt; 0$ on an interval, then __________________________.</td>
</tr>
</tbody>
</table>

Our overall goal is to be able to use this information to better understand our original function $f$.

Before we extrapolate more information about our function $f$, we will prove the Increasing/Decreasing test. But in order to do this we will need a result first—The Mean Value Theorem.

The Mean Value Theorem

The Mean Value Theorem is one of the most important theorems that we will discuss this semester.

Though the Mean Value Theorem holds for real-valued differentiable functions, it does not hold for complex-valued differentiable functions (functions with complex or (imaginary) coefficients) making it a distinguishing factor between complex and real analysis.
Mean Value Theorem

If \( f \) is a differentiable function on the interval \([a, b]\), then there exists a number \( c \) between \( a \) and \( b \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

or, equivalently,

\[f(b) - f(a) = f'(c)(b - a).\]

Let’s break down the meaning of this theorem...

(I) What are the assumptions of the Mean Value Theorem?

(II) What is the consequence of the Mean Value Theorem?

(III) To better understand what this theorem is saying,

(1) Sketch a graph of a function \( f \) on an interval \([a, b]\). (2) Then draw the secant line to this function through \( x = a \) and \( x = b \). (3) Now, find a point \( c \) on \( f \) so that the slope of the tangent line of \( f \) at \( x = c \) is parallel to the secant line through \( x = a \) and \( x = b \).

The Mean Value Theorem guarantees that such a \( c \) will always exist.

To consider a more tangible example of the Mean Value Theorem.

If a car traveled 120 miles in 2 hours, the Mean Value Theorem guarantees that the speedometer must have read 60 mph at some point during the drive.

We will now use this Mean Value Theorem to prove the Increasing/Decreasing Test.

Proof of the Increasing/Decreasing Test

We want to prove that...
(1) If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval.

and

(2) If \( f'(x) < 0 \) on an interval, then \( f \) is decreasing on that interval.

We will prove (1) together and you will prove (2) on your own.

In order to prove this theorem (and most any theorem), we will want to start by assuming the assumptions and work through a series of steps until our result is the consequence.

What is the assumption of (1)?

What is the consequence of (1)?

**proof of (1):** Assume ________________________________________________________________.

Let \( x_1 \) and \( x_2 \) be two numbers on this interval with \( x_1 < x_2 \).

Because \( f'(x) > 0 \), we know that \( f'(x) \) exists and hence \( f \) is ________________ on \([x_1, x_2]\).

So by the Mean Value Theorem,

__________________________________________________________________________.

By assumption, \( f'(c) > 0 \) and \( x_2 - x_1 > 0 \) since \( x_1 < x_2 \).

Thus \( f'(c)(x_2 - x_1) > 0 \).

Since \( f'(c)(x_2 - x_1) = f(x_2) - f(x_1) \), ________________ > 0.

This gives us ______________________________________ which means that \( f \) is increasing on the interval (according to the definition of an increasing function (p. 21 of Stewart)).

Now, the proof of (2) works almost exactly the same...

**proof of (2):**
The First and Second Derivative Tests

The Increasing/Decreasing Test and the Concavity Test will both provide us with information about the extreme values that we found in Section 4.2.

More specifically...

(a) If \( f' \) changes from positive to negative at a point \( c \), then the function \( f \) is changing from ____________________________.

Also, \( f'(c) = \)_____ and the point \( x = c \) will be a local _____________________.

(b) If \( f' \) changes from negative to positive at a point \( c \), then the function \( f \) is changing from ____________________________.

Also, \( f'(c) = \)_____ and the point \( x = c \) will be a local _____________________.

(c) If \( f' \) does not change sign at the point \( c \), then \( f \) has no local maximum or minimum at \( c \).

These criterion are what we call the First Derivative Test.

<table>
<thead>
<tr>
<th>THE FIRST DERIVATIVE TEST</th>
</tr>
</thead>
</table>
| Suppose that \( c \) is a critical number of a continuous function \( f \).
| (a) If \( f' \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \).
| (b) If \( f' \) changes from negative to positive at \( c \), then \( f \) has a local minimum at \( c \).
| (c) If \( f' \) does not change sign at \( c \), then \( f \) has no local maximum or minimum at \( c \). |

Looking at our second derivative...

(a) If \( f''(x) > 0 \) for all \( x \) on some interval containing \( c \), then

______________________________.

Also, if \( f'(c) = 0 \), this means that the point \( x = c \) will be a local _________________________.

(b) If \( f'(c) = 0 \) and \( f''(x) < 0 \) for all \( x \) on some interval containing \( c \), then

______________________________.

Also, if \( f'(c) = 0 \), this means that the point \( x = c \) will be a local _________________________.

These criterion are what we call the Second Derivative Test.

<table>
<thead>
<tr>
<th>THE SECOND DERIVATIVE TEST</th>
</tr>
</thead>
</table>
| Suppose that \( f \) is continuous near \( c \).
| (a) If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \).
| (b) If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a local maximum at \( c \). |
Exercises

For the following functions identify the following criteria.

(a) Find the intervals of increase or decrease.

(b) Find the local maximum and minimum values.

(c) Find the intervals of concavity and the inflection points.

(d) Use the information from parts (a)-(c) to sketch the graph.

1. \( f(x) = 2 + 3x - x^3 \)

2. \( h(x) = (x + 1)^5 - 3x - 2 \)
3. \( f(x) = \sqrt{x}e^{-x} \)

4. \( k(x) = x\sqrt{x + 2} \)