Limits with Indeterminate Forms

The rules presented in this section helps us evaluate limits that have indeterminate forms. These indeterminate forms have many types that all require different techniques that will be broken down in the sections that follow.

Type 1: \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \)

The first types of indeterminate form we will look at are when a limit appears to equal \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \).

Try to evaluate the following limits:

(1) \( \lim_{x \to 0} \frac{\sin x}{x} \)

(2) \( \lim_{x \to \infty} \frac{\ln x}{x-1} \)

Notice that both of these limits have indeterminate forms. when these forms arise, we can use L’HOSPITAL’S RULE.

**TRICK: L’Hospital’s Rule**

**L’Hospital’s Rule**

Suppose \( f \) and \( g \) are differentiable and \( g'(x) \neq 0 \) near \( a \) (except possibly at \( a \)). Suppose that

\[
\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0
\]

or that

\[
\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty
\]

(In other words, we have an indeterminate form of the type \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).) Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

if the limit on the right side exists (or is \( \infty \) or \( -\infty \)).
What are the *assumptions* of L’Hôpital’s Rule?

What is the *consequence* of L’Hôpital’s Rule?

While constructing a *general* proof of L’Hôpital’s Rule is quite difficult, for the case where

\[ f(a) = g(a) = 0, \quad f' \text{ and } g' \text{ are continuous and } g'(a) \neq 0, \]

the proof of L’Hôpital’s Rule is straightforward.

Let’s do walk through the proof of this specific case in L’Hospital’s Rule together. In doing so, I will provide you with all of the steps and you will write the justifications for those steps at the space provided at the right of the step.

**proof:** Assume \( f \) and \( g \) are differentiable, \( g'(x) \neq 0 \) near \( a \) (except possibly at \( a \)), \( f(a) = g(a) = 0 \), \( f' \) and \( g' \) are continuous, and \( g'(a) \neq 0 \).

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} \quad \text{since } \quad \]

\[
= \frac{f'(a)}{g'(a)} \quad \text{since } \quad \]

\[
= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{by } \quad \]

\[
= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{by } \quad \]

\[
= \lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{since } \quad \]

So \( \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)} \) which concludes our proof.
Now, let’s refer back to our original examples on page 1:

(1) \( \lim_{x \to 0} \frac{\sin x}{x} \)  
(2) \( \lim_{x \to \infty} \frac{\ln x}{x - 1} \)

Do (1) and (2) fall under the assumptions of L’Hospital’s Rule?

If so, apply the consequence of L’Hospital’s Rule to evaluate the limits.

(1) \( \lim_{x \to 0} \frac{\sin x}{x} \)

(2) \( \lim_{x \to \infty} \frac{\ln x}{x - 1} \)

Note that L’Hospital’s Rule can also be applied to left and right handed limits as well as infinite limits.

More Examples

Calculate the following limits.

1. \( \lim_{x \to 1} \frac{x^2 - 1}{x^2 - x} \)

2. \( \lim_{x \to \infty} \frac{e^{x/10}}{x^3} \)
3. \( \lim_{x \to \pi} \frac{\sin x}{1 - \cos x} \)

**Type 2: \( 0 \cdot \infty \) and \( 0 \cdot -\infty \)**

The second type of indeterminate form we will look at are when a limit appears to equal \( 0 \cdot \infty \) and \( 0 \cdot -\infty \).

Try to evaluate the following limit:

\[
\lim_{x \to -\infty} x^2 e^x
\]

Notice that it has the indeterminate form \( \infty \cdot 0 \).

Here, our plan of attack will be to try to rewrite this in a form so it fits the criteria for L’HOSPITAL’S RULE.

**TRICK: Write** \( fg = \frac{f}{1/g} \) or \( fg = \frac{g}{1/f} \).

Our trick will be to use the fact that \( fg = \frac{f}{1/g} \).

Verify for yourself that the left and right hand sides of this equation are in fact equal.

For our example, \( x^2 e^x \), what are \( f \) and \( g \)?

Rewrite \( \lim_{x \to -\infty} x^2 e^x \) as \( \lim_{x \to -\infty} \frac{f}{1/g} \) and use L’Hospital’s Rule to evaluate the limit.
Type 3: $\infty - \infty$

The third type of indeterminate form we will look at are when a limit appears to equal $\infty - \infty$.

Try to evaluate the following limit:

$$\lim_{x \to 0^+} \csc x - \cot x$$

Notice that it has the indeterminate form $\infty - \infty$.

**TRICK:** Try to get a common denominator.

Get a common denominator on $\csc x - \cot x$ and use L’Hospital’s Rule to find $\lim_{x \to 0} \csc x - \cot x$.

Type 4: $0^0$, $\infty^0$ and $1^\infty$

The final type of indeterminate form we will look at are when a limit appears to equal $0^0$, $\infty^0$ or $1^\infty$.

Try to evaluate the following limit:

$$\lim_{x \to 0^+} x^x$$

Notice that it has the indeterminate form $0^0$.

**TRICK:** Rewrite in terms of natural logarithms.

Our trick is to use the fact that $f^g = e^{g \ln f}$.

Verify for yourself that the left and right hand sides of this equation are in fact equal.

For our example, $x^x$, what are $f$ and $g$?
Rewrite \( \lim_{x \to 0^+} x^x \) as \( \lim_{x \to 0^+} e^{x \ln x} \) and use L’Hospital’s Rule to evaluate the limit.

Exercises

Find the following limits where possible.

1. \( \lim_{x \to 0} \frac{\sin 4x}{\tan 5x} \)

2. \( \lim_{x \to 1} \frac{\ln x}{\sin \pi x} \)

3. \( \lim_{x \to \infty} \sqrt{x^2 + x} - x \)
4. \( \lim_{x \to \infty} x^{1/x} \)

5. \( \lim_{x \to \infty} x \sin(\pi/x) \)

6. \( \lim_{x \to 0^+} \left( \cot x - \frac{1}{x} \right) \)