The Substitution Rule

Warm-Up
Evaluate the following definite integrals:

1. \( \int_0^1 x^{4/5} \, dx \)

2. \( \int_{-5}^{5} e \, dx \)

3. \( \int_{-1}^{1} t(1 - t)^2 \, dt \)

4. \( \int_1^2 \frac{v^3 + 3v^6}{v^4} \, dv \)

The Indefinite Integral
The information that we have at this point is sufficient to compute the previous integrals. However, it is not too difficult to imagine a function that we would be unable to integrate at this point.

For instance,

\[ \int 2x\sqrt{1 + x^2} \, dx. \]

We cannot algebraically simplify \( 2x\sqrt{1 + x^2} \) like in the last two Warm-Up exercises. So instead, we will introduce a new variable.

In this particular example, our variable that we will want to introduce is \( u = 1 + x^2 \).
Now, find the differential of this new variable $u = 1 + x^2$.

Referring back to our integral, notice that

$$\int 2x\sqrt{1 + x^2} \, dx = \int \sqrt{1 + x^2} \cdot 2x \, dx = \int \sqrt{u} \, du.$$  

Now, integrate $\int \sqrt{u} \, du$.  

Note: Don’t forget the ‘$+C$’ since this is an indefinite integral!

Finally, substitute $u = 1 + x^2$ back in for $u$.

That is your integral!  
Check to make sure that your integration is correct.

Notice that this technique (often referred to as ‘u-substitution’) can be thought of as the integration equivalent to the Chain Rule.

**The Substitution Rule**

If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ and continuous on $I$, then

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du$$
Examples

Evaluate the following indefinite integrals:

1. \( \int x \sin(x^2) \, dx \)

2. \( \int \frac{x}{(x^2+1)^2} \, dx \)

3. \( \int e^x \cos(e^x) \, dx \)

The Definite Integral

Since we have already computed \( \int 2x\sqrt{1+x^2} \, dx \), we know \( F(x) \) (an antiderivative of \( 2x\sqrt{1+x^2} \)).

\[
\int_{-1}^{2} 2x\sqrt{1+x^2} \, dx.
\]

Since we have already computed \( \int 2x\sqrt{1+x^2} \, dx \), we know \( F(x) \) (an antiderivative of \( 2x\sqrt{1+x^2} \)).

We have two options for how we want to compute the definite integral...

Option 1:
We can use the *Evaluation Theorem* (or Part 2 of the Fundamental Theorem of Calculus) in order to figure out how to apply u-substitution to evaluate definite integrals.
Option 2:
Use the Substitution Rule for Definite Integrals.

**The Substitution Rule for Definite Integrals**

If \( g' \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( u = g(x) \), then

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

**Examples**

Evaluate the following definite integrals:

1. \( \int_0^1 \cos(\pi t/2) \, dt \)

2. \( \int_0^1 \sqrt{1 + 7x} \, dx \)
3. \( \int_{1}^{e} \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \)

4. \( \int_{0}^{1} \frac{e^{x}+1}{e^{x}+z} \, dz \)

5. \( \int_{e}^{e^{4}} \frac{dx}{x \ln x} \)

6. \( \int_{\pi/2}^{\pi} \cos x \sin(\sin x) \, dx \)