The Mathematics of Power: Weighted Voting

The Electoral College offers a classic illustration of weighted voting. The Electoral College consists of 51 ‘voters’ (the 50 states plus DC) each with a weight determined by the Congressional delegation (number of Representative and Senators). At one end of the spectrum is heavily weighted California (55 electoral votes) and at the other end are the small states like Wyoming, Montana, North Dakota and DC (with 3 electoral votes) (see map on p.36).

Since the point of weighted voting is to give different voters different amounts of influence, the key question we are concerned with in this chapter is: How does one go about measuring a voter’s power in a weighted voting situation?

2.1 An Introduction to Weighted Voting

So what is weighted voting?

Unlike in chapter 1 where the discussion focused primarily on elections involving three or more choices, in this chapter we will consider voting on yes-no votes, known as motions. Every weighted voting system is characterized by three elements:

- players:
- weights:
- quota:

Thus a generic weighted voting system with $N$ players can be written as:

Examples

1. Venture Capitalism: Five partners ($P_1$, $P_2$, $P_3$, $P_4$ and $P_5$) decide to start a new business venture. In order to raise the $200,000 venture capital needed for startup money, they issue 20 shares worth $10,000 each. Suppose that $P_1$ buys 8 shares, $P_2$ 5 shares, $P_3$ buys 3 shares, $P_4$ buys 2 shares and $P_5$ buys 2 shares with the usual agreement that one share equals one vote in the partnership.

Suppose the quota is set to be a two-thirds of the total number of votes. What is $q$?
Suppose now that instead the quota is 10 votes. What is the problem with this system?

Suppose instead that the quota is 21 votes. What is the problem with this system?

Is there any way to choose the quota so that neither a gridlock nor anarchy will occur?

Finally, suppose the quota is 19 votes. Why is this system interesting?

2. Consider the weighted voting system \([30 : 10, 10, 10, 9]\). What is the problem with this system?

3. Consider the weighted voting system \([12 : 13, 6, 4]\). What is the problem with this system?

4. Consider the weighted voting system \([12 : 9, 5, 4, 2]\). Why is this system interesting?

2.2 Banzhaf Power

There is an important lesson that can be drawn from the preceding examples:

To pursue this further, we need a formal definition of ‘power’ and how it can be measured. The method we will discuss was first proposed in 1965 by a law Professor John Banzhaf III.

Example: US Senate

The US Senate has 100 members and a simple majority of 51 votes is required to pass a bill. Suppose that the Senate is composed of 49 Republicans, 48 Democrats and 3 Independents and that all Senators vote strictly along party lines. What type of weighted voting system is this?
Analyze the weighted voting system.

Before we continue, let’s introduce some important new concepts:

- **coalition:**

- **winning coalition:**

- **critical player:**

- **critical count:**

Analyze the US Senate example in terms coalitions, critical players and critical count.

Now let's turn to the key concept of this section—the **Banzhaf power index (BPI)**.
Computing the Banzhaf Power Distribution

Step 1 Make a list of the winning coalitions.
Step 2 Within each winning coalition determine which are the critical players (i.e. underline them).
Step 3 Find the critical counts $B_1, B_2, ..., B_N$.
Step 4 Find $T = B_1 + B_2 + ... + B_N$.
Step 5 Compute the BPI's: $\beta_1 = \frac{B_1}{T}, \beta_2 = \frac{B_2}{T}, ..., \beta_N = \frac{B_N}{T}$

Examples

**Example:** [4 : 3, 2, 1]
Find the Banzhaf power distribution of the weighted voting system [4 : 3, 2, 1].

Banzhaf introduced the concept of BPI in the following legal dispute. Essentially, Banzhaf argued that it’s the critical count and not the weight, that truly measures a player’s power.

If the critical count of $X$ is the same as that of $Y$ then $X$ and $Y$ have equal power. If the critical count of $X$ is double that of $Y$ then $X$ have twice as much power as $Y$.

**Example:** Nassau County (NY) Board of Supervisors (1960s)
Throughout the 1900s county boards in NY operated as weighted voting systems. The reasoning behind weighted voting was that counties are often divided into districts of uneven size and it seemed unfair to give an equal vote to both large and small districts. To eliminate the unfairness a system of proportional representation was used: Each district would have a number of votes roughly proportional to its population. Every 10 years, after the Census, the allocation of votes could change if the population changed but the principle of the proportional voting remained.

Nassau County was divided into 6 districts: Hempstead 1 (H1) with 31 votes, Hempstead 2 (H2) with 31 votes, Oyster Bay (OB) with 28 votes, North Hempstead (NH) with 21 votes, Long Beach (LB) with 2 votes and Glen Cover (GC) with 2 votes. The total number of votes $V = 115$ and the quota $q = 58$.
Determine the Banzhaf Power Distribution.

This example reinforces the fact that in a weighted voting you can have a lot of votes and no power. A player with votes but zero power is called a dummy.
2.3 Shapley-Shubik Power

A different approach to measuring power was proposed by American mathematician Lloyd Shapley and economist Martin Shubik in 1954. The key difference between the Shapley-Shubik measure of power and the Banzhaf measure of power centers on the concept of *sequential coalition*.

To illustrate this concept, let’s revisit the [4 : 3, 2, 1] example. Suppose the players join coalitions one at a time and that we want to consider the order in which the players join the coalition. Let’s observe what happens:

Let’s now formally define some of the concepts presented in this example.

- **sequential coalition:**

- **pivotal player:**

- **pivotal count:**

- **Shapley-Shubik power index (SSPI):**

- **Shapley-Shubik power distribution:**

<table>
<thead>
<tr>
<th>Computing the Shapley-Shubik Power Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1 Make a list of the $N!$ sequential coalitions with $N$ players.</td>
</tr>
<tr>
<td>Step 2 In each sequential coalition determine <em>the</em> pivotal player (i.e. underline them).</td>
</tr>
<tr>
<td>Step 3 Find the <em>pivotal counts</em> $SS_1, SS_2, ..., SS_N$.</td>
</tr>
<tr>
<td>Step 4 Compute the SSPIs: $\sigma_1 = \frac{SS_1}{M}$, $\sigma_2 = \frac{SS_2}{M}$, ..., $\sigma_1 = \frac{SS_N}{M}$.</td>
</tr>
</tbody>
</table>
Example: NBA Draft
Find the Shapley-Shubik power distribution of the weighted voting system [6 : 4, 3, 2, 1].

Compare this to the Banzhaf power distribution computed in Example 2.10 on p.44 of the text. Do you think this comparison will hold true in general?

Compare Example 2.14 (p.48) and 2.18 (p.52) in the text and discuss the results.

2.4 Subsets and Permutations

The purpose of this section it to provide a quick introduction to two very important mathematical concepts: subsets and permutations.

Subsets and Coalitions

A subset of a set is any combination of elements from the set. This includes the set with nothing in it i.e. the empty set.

List all of the subsets of \{P_1, P_2\}.

List all of the subsets of the set \{P_1, P_2, P_3\}.

List all of the subsets of the set \{P_1, P_2, P_3, P_4\}.

What do you notice? Can you make any generalizations?
**Facts**

- A set with $N$ elements has ________ subsets.
- A weighted voting system with $N$ players has ________ coalitions.
- A weighted voting system with $N$ players has ________ coalitions of 2 or more players.

**Permutations**

A **permutation** of a set of objects is an ordered list of the objects.

List all of the permutations of $\{P_1, P_2\}$.

List all of the permutations of the set $\{P_1, P_2, P_3\}$.

List all of the permutations of the set $\{P_1, P_2, P_3, P_4\}$.

What do you notice? Can you make any generalizations?

**Facts**

- A set with $N$ elements has ________ different permutations.
- A weighted voting system with $N$ players has ________ sequential coalitions.
- Let $P$ be a player in a weighted voting system with $N$ players and $k$ an arbitrary position between first and last. There are ________ sequential coalitions with $P$ in the $k$th position.

**Homework:** Ch.2 (p.59) # 2, 4, 6, 8, 10, 12, 16, 18, 20, 30, 32, 40, 44, 55, 64