When different parties—with different preferences and values—have to divide commonly owned assets, it is actually possible to carry out the division in a way so that each party comes out of the deal feeling that they got more than they deserved.

A *fair division problem* is one in which a set of assets is to be divided among a group of individuals that have some equity in the assets and a shared goal of dividing the assets a fair way. In this chapter we will try to answer the question:

> What are the elements of a fair-division game and what are the requirements for a fair-division method?

### 3.1 Fair-Division Games

The basic elements of a fair-division game are:

- **assets**
- **players**
- **value systems**
- **fair-division method**

Fair-division games are predicated on four assumptions

<table>
<thead>
<tr>
<th>rationality</th>
<th>cooperation</th>
<th>privacy</th>
<th>symmetry</th>
</tr>
</thead>
</table>

Given a set of players $P_1, P_2, ..., P_N$ and a set of assets $S$, the purpose of a fair-division game is to produce a fair division of $S$ but the question remains: What does this *mean*?

To address this question we need to introduce two definitions:

- **fair share**:
- **fair division**: 

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The Mathematics of Sharing: Fair Division Games
Let’s examine the meaning of these terms with an example.

**Example:** Three brothers–Henry, Tom and Fred–are splitting up their partnership and dividing up assets of unspecified value. The set of assets $S$ is divided into three shares $s_1$, $s_2$ and $s_3$. In Henry’s opinion the values of the three shares are 32%, 31% and 37% respectively. For Tom, the values of the three shares are 34%, 31% and 35% respectively and for Fred, the values of the three shares are 33.3%, 33.3% and 33.3% respectively.

What are the fair shares for each player?

What are the fair divisions?

**Fair-Division Methods**

A **fair-division method** is a set of rules that, when properly used by the players, guarantees that at the end of the game each player will have received a fair share of the assets.

Come up with a reasonable method for the determining the distribution of the assets in the above example that fails to be a **fair-division method**.

There are many different fair division methods known but we will discuss only a few. Depending on the nature of the set $S$, a fair-division game can be classified as follows:

- **continuous**
- **discrete**
- **mixed**

*Mixed fair division games* can usually be solved by dividing the discrete and continuous parts separately so we will not discuss them in detail. We will begin our examination with a classic method for continuous fair division.

### 3.2 The Divider-Chooser Method

When two players are dividing a continuous asset, the standard method used is the **divider-chooser method**. As the name suggests, one player, called the *divider*, divides $S$ into two shares and the second player, called the *chooser*, picks the share he or she wants, leaving the other share to the divider.

Note that this uses the *privacy* and *rationality* assumptions above. Not knowing the chooser’s likes and dislikes (privacy assumption), the divider can only guarantee himself a 50% share by dividing $S$ into two halves of equal value (rationality assumption); the chooser is guaranteed a 50% or better share by choosing the piece she likes best.

When using the divider-chooser method, do you think it is better to be the *divider* or the *chooser*? Why?
But what if there are more than two players?

3.3 The Lone-Divider Method

The first important breakthrough in the mathematics of fair division came in 1943, when the Polish mathematician Hugo Steinhaus came up with a clever way to extend some of the ideas of the divider-chooser method to the case of three players—one plays the divider and two play the choosers. Princeton mathematician Harold Kuhn generalized Steinhaus’ approach to account for any number (N—one divider and N − 1 choosers) players. This method is known as the lone-divider method.

The Lone-Divider Method for Three Players

Preliminaries: One of the three players will be the divider, $D$; the other two players are the choosers, $C_1$ and $C_2$. Since it is better to be a chooser than a divider, the decision is made by random draw.

Step 1: Division: The divider $D$ divides the assets $S$ into three shares, $s_1, s_2, s_3$. $D$ will get one of these shares but at this point does not know which one. Not knowing which share will be his forces $D$ to divide $S$ into shares of equal value.

Step 2: Bidding: $C_1$ declares (usually on a slip of paper which of the three pieces are fair shares to her. Independently, $C_2$ does the same. These are called bids.

Step 3: Distribution: We will separate the bids into two types: $C$-pieces (the pieces chosen by either one or both choosers) and $U$-pieces (the unwanted pieces that did not appear in either of the bids).

Depending on the number of $C$-pieces there are two cases to consider:

3A: When there are two or more $C$ pieces there is always a way to give each chooser a different piece from among the pieces listed in her bid. Once each chooser gets her piece, the divider gets the last remaining piece. At this point every player has received a fair share.

3B: When there is only one $C$-piece, we have a bit of a problem because it means that both choosers are bidding for the same piece. First, we take care of the divider, $D$ by giving him one of the pieces neither chooser wants. After $D$ gets his piece, the two pieces left are recombined into one piece that we call the $B$-piece. We now have one piece and two players to which we revert to the divider-chooser method.

Let’s examine an example. Bear in mind that the following information is never available in full to the players—an individual only knows his or her own preferences.

Example: Dale, Cindy and Cher are dividing a cake using Steinhaus’s lone-divider method. They draw cards from a well-shuffled deck of cards and Dale draws the low card and has to be the divider. Consider the following variations.

1. Step 1: Division: Dale divides $S$ into three pieces $s_1, s_2, s_3$ of equal value (in his eyes).

   Step 2: Bidding: Cindy and Cher also assign values to each bid as follows:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dale</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cindy</td>
<td>35%</td>
<td>10%</td>
<td>55%</td>
</tr>
<tr>
<td>Cher</td>
<td>40%</td>
<td>25%</td>
<td>35%</td>
</tr>
</tbody>
</table>

What are Cindy and Cher’s bid lists?
Step 3: DISTRIBUTION: What are the distributions based on the lone-divider method?

2. Step 1: DIVISION: Dale divides $S$ into three pieces $s_1, s_2, s_3$ of equal value (in his eyes).
Step 2: BIDDING: Cindy and Cher also assign values to each bid as follows:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dale</td>
<td>33.3%</td>
<td>33.3%</td>
<td>33.3%</td>
</tr>
<tr>
<td>Cindy</td>
<td>30%</td>
<td>40%</td>
<td>30%</td>
</tr>
<tr>
<td>Cher</td>
<td>60%</td>
<td>15%</td>
<td>25%</td>
</tr>
</tbody>
</table>

What are Cindy and Cher’s bid lists?

Step 3: DISTRIBUTION: What are the distributions based on the lone-divider method?

3. Step 1: DIVISION: Dale divides $S$ into three pieces $s_1, s_2, s_3$ of equal value (in his eyes).
Step 2: BIDDING: Cindy and Cher also assign values to each bid as follows:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dale</td>
<td>33.3%</td>
<td>33.3%</td>
<td>33.3%</td>
</tr>
<tr>
<td>Cindy</td>
<td>20%</td>
<td>30%</td>
<td>50%</td>
</tr>
<tr>
<td>Cher</td>
<td>10%</td>
<td>20%</td>
<td>70%</td>
</tr>
</tbody>
</table>

What are Cindy and Cher’s bid lists?

Step 3: DISTRIBUTION: What are the distributions based on the lone-divider method?

The Lone-Divider Method for More than Three Players

How do you think that this method can be extended to account for more than three players?
3.4 The Lone-Chooser Method

A completely different approach for extending the divider-chooser method was proposed by A. M. Fink (a mathematician at Iowa State University) in 1964. It is known as the lone-chooser method since in this method one player plays the role of chooser and all other players start out playing the role of dividers.

The Lone-Chooser Method for Three Players

**Preliminaries:** One of the three players will be the chooser, \( C \); the other two players are the dividers, \( D_1 \) and \( D_2 \). Since it is better to be a chooser than a divider, the decision is made by random draw.

**Step 1:** Division: Using the divider-chooser method, the dividers \( D_1 \) and \( D_2 \) divide the assets \( S \) between themselves into two fair shares, \( s_1 \) and \( s_2 \).

**Step 2:** Subdivision: Each player divides his share into three shares: \( D_1 \) divides \( s_1 \) into \( s_{1a}, s_{1b}, s_{1c} \) and \( D_2 \) divides \( s_2 \) into \( s_{2a}, s_{2b}, s_{2c} \).

**Step 3:** Selection: Chooser \( C \) now selects one of \( D_1 \)'s three shares and one of \( D_2 \)'s three shares.

**Why is this a fair division of \( S \)? (i.e. What is the value of each person’s share?)**

**Example:** David, Dinah and Cher are dividing the orange-pineapple cake using the lone-chooser method. The cake is valued at $27, so each of them expects to end up with a share worth at least $9. Their individual value systems (not known to the other players are as follows:

<table>
<thead>
<tr>
<th></th>
<th>pineapple</th>
<th>orange</th>
</tr>
</thead>
<tbody>
<tr>
<td>David</td>
<td>$13.50</td>
<td>$13.50</td>
</tr>
<tr>
<td>Dinah</td>
<td>$0</td>
<td>$27</td>
</tr>
<tr>
<td>Cher</td>
<td>$18</td>
<td>$9</td>
</tr>
</tbody>
</table>

After random selection, Cher gets to be the chooser.

**Step 1:** Division: David and Dinah start dividing the cake between them. After a coin flip, David cuts the cake into two pieces as shown: Which piece will Dinah choose?

**Step 2:** Subdivision: How do you think David and Dinah will divide their shares?

**Step 3:** Selection: How will Cher choose her shares from David and Dinah?
The Lone-Chooser Method for More than Three Players

How do you think that this method can be extended to account for more than three players?

The next two sections will discuss discrete fair-division methods—methods for dividing assets consisting of indivisible objects such as houses, cars, art, etc. As a general rule, discrete fair division is harder to achieve than continuous fair division because there is less flexibility in the division process.

3.5 The Method of Sealed Bids

The method of sealed bids was originally proposed in 1948 by Polish mathematicians Steinhaus and Bronislaw Knaster. The best way to illustrate the method is by example.

**Example: Selling Grandma’s Estate**

In her will and testament, Grandma plays a little joke on her four grandchildren (Art, Betty, Carla and Dave) by leaving them just three items—a cabin in the mountains, a vintage 1955 Rolls Royce and a Picasso painting—with the stipulation that the items must remain with the grandchildren (not sold to outsiders) and must be divided fairly among them.

The method of sealed bids will give us an elegant solution to this problem.

**Step 1: Bidding:** Each of the players makes a bid (in dollars) for each of the items in the estate, giving his or her honest assessment of the value of each item. These bids are done independently.

<table>
<thead>
<tr>
<th></th>
<th>Art</th>
<th>Betty</th>
<th>Carla</th>
<th>Dave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cabin</td>
<td>420,000</td>
<td>450,000</td>
<td>411,000</td>
<td>398,000</td>
</tr>
<tr>
<td>Vintage Rolls</td>
<td>80,000</td>
<td>70,000</td>
<td>87,000</td>
<td>92,000</td>
</tr>
<tr>
<td>Painting</td>
<td>680,000</td>
<td>640,000</td>
<td>634,000</td>
<td>590,000</td>
</tr>
<tr>
<td>Total Value</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Step 2: Allocation:** Each item will go to the highest bidder for that item. (If there is a tie, the tie will be broken with a coin flip.)

Who gets what?

**Step 3: First Settlement:** Now it is time to settle things up. Depending on what items (if any) a player gets in Step 2, he or she will owe money to or be owed money by the estate. To determine how much money a player owes or is owed, we first calculate the fair-share value of the estate. A player’s **fair share value** is found by adding the players bids and dividing the total by the number of players.

The fair-share values are the baseline for the settlements—if the total value of the items that the player gets in Step 2 is more than his or her fair-share value then the player pays the estate difference. If the total value of the items that the player gets is less than his or her fair share then the player gets the difference in cash.
What happens if you add Art and Betty’s payments to the estate and subtract the payments made by the estate to Carla and Dave?

Step 4: Division of Surplus: The surplus is the common money that is added to the estate and thus must be divided among the players.

What is each player’s share of the surplus?

Step 5: Final Settlement: What is the final outcome?

3.6 The Method of Markers

The method of markers is a discrete fair-division method proposed in 1975 by William F. Lucas, a mathematician at Claremont Graduate School. This method has the virtue of not requiring the players to put up any of their own money. However, this method can only be used effectively in the case of a fine-grained, discrete, fair division game.

Preliminaries: Items are arranged randomly into an array. For convenience, label the items 1 through M from left to right. Assume there are N players.

Step 1: Bidding: Each player independently divides the array into N segments by placing N – 1 markers along the array. These segments are assumed to represent fair shares of the array in the opinion of that player.

Step 2: Allocations: Scan the array from left to right unit the first marker is located. The player owning that marker (call her P₁) goes first and gets the first segment in her bid. (In case of a tie, break the tie randomly.) P₁’s markers are removed and we continue scanning from left to right looking for the first second marker. The player owning that marker (call him P₂) goes second and gets the second segment in his bid. Continue this process. The last player gets the last segment in her bid.

Step 3: Dividing the Surplus: The players take turns in some random order and pick one at a time until the surplus items are given out.

Example: Dividing Valentine’s Candy

<table>
<thead>
<tr>
<th>Item(s) received</th>
<th>Art</th>
<th>Betty</th>
<th>Carla</th>
<th>Dave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value received</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fair-share value</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To (from) estate</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>