NAME ______________________________

The Mathematics of Apportionment

Representatives . . . shall be apportioned among the several States . . . according to their respective Numbers. The actual Enumeration shall be made . . . every ten Years . . .
- Article 1, Section 2, US Constitution

The constitution mandates that every 10 years (years ending in ‘0’) the government produce a ‘head count’ broken down by state. The key purpose of the state population numbers is to meet the constitutional requirement of ‘proportional representation.’ The Constitution requires that seats in the House of Representatives be “apportioned among the . . . States according to their respective numbers” so every 10 years, two things happen:

1. the state population must be determined (by Census) and
2. the seats in the House of Representatives must be apportioned to the states based on their populations.

While this does not sound all that complicated, we will investigate in this chapter the mathematics behind appointment and come to find that it has some interesting features.

We will begin with an investigation into the following questions:

What is an apportionment problem? and What is an apportionment method?

4.1 Apportionment Problems and Apportionment Methods

There are two key elements of the dictionary definition of the word apportion:

To get an idea of where interesting issues might arise with apportionment problems, let’s look at an example.

Example: Kitchen Capitalism

Mom has a total of 50 identical caramels which she is planning to divide among her five children. She could clearly divide them evenly by giving each child 10 caramels but she wants to teach them a lesson about the value of work and the relationship between work and reward so instead…

She announces to the kids that the candy is going to be divided at the end of the week in proportion to the amount of time each of them spends helping with the weekly kitchen chores (i.e. if you worked twice as long as your brother, you get twice as much candy).

The following table shows the amount of work (in minutes) that each child completed during the week:

<table>
<thead>
<tr>
<th>Child</th>
<th>Alan</th>
<th>Betty</th>
<th>Connie</th>
<th>Doug</th>
<th>Ellie</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minutes worked</td>
<td>150</td>
<td>78</td>
<td>173</td>
<td>204</td>
<td>295</td>
<td></td>
</tr>
</tbody>
</table>

According to the rules, how much candy should each child get?
Apportionment: Basic Concepts and Terminology

While there are many different examples of appointment (assigning classrooms to departments in a university, assigning nurses to shifts in a hospital, etc), the standard practice in discussing appointment methods is to borrow terminology from that of legislative appointment and apply it to apportionment problems in general.

- states:
- seats:
- populations:
- apportionment method:

There are also three important concepts that relate to apportionment methods:

**standard divisor (SD):**

**standard quotas:**

**upper and lower quotas:**

Let’s now introduce a fictional (but realistic) example that we will return to a few times.

**Example: The Congress of Power**

Parador is a small republic located in central America and consists of six states: Azcar, Bahia, Café, Diamante, Esmeralda, Felicidad (A, B, C, D, E and F). There are 250 seats in the Congress which are to be apportioned among the six states to their respective population. The following table shows the populations based on the most recent census:

<table>
<thead>
<tr>
<th>State</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>1,646,000</td>
<td>6,936,000</td>
<td>154,000</td>
<td>2,091,000</td>
<td>685,000</td>
<td>988,000</td>
<td>12,500,000</td>
</tr>
</tbody>
</table>

Compute the **standard divisor (SD).**

Compute the **standard quotas.**

Standard quotas are the benchmark for *fair apportionment*, however we need to somehow convert them to whole numbers (because we cannot have fractions of people!). Rounding is probably the most natural way to do this, but there is a problem. What is it?

The most important moral of this example is that

*conventional rounding of standard quotas is not an apportionment method*.

since it does not *guarantee* that it will apportion *M* seats. Our search of a good solution to this problem is the goal of the rest of the chapter.
4.2 Hamilton’s Method

In 1792 Alexander Hamilton proposed the following simple method as a way to apportion the US House of Representatives.

**Hamilton’s Method**

| Step 1: Calculate each state’s lower quota. |
| Step 2: Round the standard quota’s down and give to each state its lower quota. |
| Step 3: Give the surplus seats (one at a time) to the states with the largest residues (fractional parts) until there are no more surplus seats. |

**Example: Parador’s Congress**

Revisiting the above example, use Hamilton’s method to find an apportionment for Parador’s congress.

<table>
<thead>
<tr>
<th>State</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Quota</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower Quota</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Residue</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Appropriation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Do you notice any issues with Hamilton’s method?

4.3 Jefferson’s Method

Notice that in Hamilton’s method, it is the handling of the surplus seats that yields preferential treatment. So the idea behind our next method is to tweak things to that when the quotas are rounded down there are no surplus seats!

How is this done? By changing the divisor, we will then change the quotas. By using a smaller divisor we will make the quotas bigger. In fact, we want to get the divisor just right so that when we round down, the surplus seats are gone.

Thomas Jefferson proposed this method for the US House of Representatives in 1792 and it generally works as follows:

**Jefferson’s Method**

| Step 1: Find a ‘suitable’ divisor \(d\). |
| Step 2: Using \(d\) as the divisor, compute each state’s modified quota (modified quote = state population/\(d\)). |
| Step 3: Each state is apportioned its modified lower quota. |

The difficult part of implementing Jefferson’s method is Step 1 (i.e. picking the exact right \(d\)). Although it sounds complicated we will really only use a guess-and-check method to find such \(d\) as illustrated by the following flow chart:
Let’s now try this method out on our previous example.

**Example: Parador’s Congress**

Recall that we want to apportion 250 seats in Parador’s Congress and that we found SD to be 50,000.

In order to execute *Jefferson’s method* we must first find a new \(d\) that fits the specifications above (the flow chart). So we may start by choosing \(d < 50,000\) to be whatever we want. You may choose 49,999 or 49,000 or even 10 ... and then follow the directed computations of the flow chart.

In the interest of time, the *modified divisor* you should eventually get (or be lucky enough to choose off the bat!) will be \(d = 49,500\).

Continue on by using the second two steps of *Jefferson’s method* to find an apportionment for Parador’s congress.

<table>
<thead>
<tr>
<th>State</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>1,646,000</td>
<td>6,936,000</td>
<td>154,000</td>
<td>2,091,000</td>
<td>685,000</td>
<td>988,000</td>
<td>12,500,000</td>
</tr>
<tr>
<td>Modified Quota (d=49,500)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rounded down to</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One question that is lurking in the background is:

Can we really substitute the standard divisor by some other random divisor?

**4.4 Adam’s & Webster’s Methods**

**Adam’s Method**

As you may have guessed at this point, Adam’s method we first proposed by John Quincy Adams (in 1832) as an alternative to Jefferson’s method. Adam’s method is the mirror image of Jefferson’s: Instead of rounding all quotas down to their lower quotas, it rounds them up to their *upper quotas*. 
Notice though that in this case, something about the divisor $d$ will need to change. What is it?

**ADAM’S METHOD**

Step 1: Find a ‘suitable’ divisor $d$.
Step 2: Using $d$ as the divisor, compute each state’s modified quota (modified quota = state population/$d$).
Step 3: Each state is apportioned its modified *upper quota*.

To find $d$:

**Start:**
Guess $d$ ($d > SD$).

**Computation:**
1. Divide state populations by $d$.
2. Round numbers up.
3. Add numbers. Let total = $T$.

Make $d$ larger.

Recall that we want to apportion 250 seats in Parador’s Congress and that we found $SD$ to be 50,000.

Use *Adam’s method* to find an apportionment for Parador’s congress.

<table>
<thead>
<tr>
<th>State</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>1,646,000</td>
<td>6,936,000</td>
<td>154,000</td>
<td>2,091,000</td>
<td>685,000</td>
<td>988,000</td>
<td>12,500,000</td>
</tr>
<tr>
<td>Modified Quota ($d=50,700$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rounded up to</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Webster’s Method**

What is the obvious compromise between rounding all the quotas down (Jefferson’s method) and rounding all the quotas up (Adam’s method)?
Recall that we have tried conventional rounding of standard quotas previously but that is not an apportionment method. However, we have not yet tried using conventional rounding of modified quotas. Now that we know we can use modified divisors to manipulate the quotas, it is always possible to find a suitable divisor that will make conventional rounding work. Daniel Webster first proposed this method in 1832 so this method is called Webster’s method.

**Webster’s Method**

Step 1: Find a ‘suitable’ divisor $d$.

Step 2: Using $d$ as the divisor, compute each state’s modified quota (modified quota = state population/$d$).

Step 3: Find the apportionments by rounding each modified quota to the nearest integer.

The following is a flow chart illustrating how to find $d$ for this method.

Let’s now try this method out on our previous example.

**Example: Parador’s Congress**

Recall that we want to apportion 250 seats in Parador’s Congress and that we found SD to be 50,000.

Use the above chart to find the appropriate $d$ for Webster’s method.

Use Webster’s method to find an apportionment for Parador’s congress.
4.5 Huntington-Hill Method

Until 1940, the method used to apportion the US House of Representatives was chosen by Congress every 10 years, typically for political reasons and with little consideration given to the mathematical subtleties of apportionment or to its consistency. Hamilton’s, Jefferson’s and Webster’s were all used at some point.

In 1941, Congress passed and President Roosevelt signed “An Act to Provide for Apportioning Representatives in Congress among Several States by the Equal Proportions method” known as the 1941 apportionment act. This act made two important changes in the process of apportioning the seats in the House of Representatives every 10 years:

1. the number of seats is permanently set at \( M = 435 \), and
2. the method of apportionment is permanently set to be the Huntington-Hill method.

The Huntington-Hill method is a sophisticated variation of Webster’s method.

The Geometric Mean and the Huntington-Hill Rule

Under the Huntington-Hill method, quotas are rounded in a manner that is very similar to but not quite the same as the conventional rounding used in Webster’s method. Under the Huntington-Hill method, the cutoff point for rounding a quota down or up is a bit top the left of 0.5 and it varies with the size of the quota.

To understand these unusual cutoff points for rounding we introduce the two key concepts in this section:

- geometric mean:
- Huntington-Hill rounding rule:

**Examples:** Round \( q \) using the Huntington-Hill rounding rule.

1. \( q = 1.51 \)
2. \( q = 1.41 \)
3. \( q = 1.42 \)
4. \( q = 3.48 \)
5. \( q = 5.48 \)
6. \( q = 6.48 \)
7. \( q = 42.49 \)

You may have observed two useful facts about the Huntington-Hill rounding method:

There is only a narrow window to the left of 0.5 where Huntington-Hill rounding is different from conventional rounding.
The Huntington-Hill Method

Once we understand the Huntington-Hill rounding rule, the description of the Huntington-Hill method is straightforward.

**Huntington-Hill Method**

**Step 1:** Find a ‘suitable’ divisor $d$. [Here a suitable divisor means a divisor that produces an apportionment of exactly $M$ seats when the quotas (populations divided by $d$) are rounded using the Huntington-Hill rounding rule.]

**Step 2:** Find the apportionment of each state by rounding its quota using the Huntington-Hill rounding rule.

To find $d$:

**Start:**

Guess $d$ ($d = SD$).

**Computation:**

1. Divide state populations by $d$.
2. Round numbers using HH Rounding Rule.
3. Add numbers. Let total = $T$.

**End.**

**Make $d$ larger.**

**Make $d$ smaller.**

**Example: Parador’s Congress**

Recall that we want to apportion 250 seats in Parador’s Congress and that we found $SD$ to be 50,000.

Use the Huntington-Hill method to find an apportionment for Parador’s congress.

<table>
<thead>
<tr>
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<td>685,000</td>
<td>988,000</td>
<td>12,500,000</td>
</tr>
<tr>
<td>Modified Quota</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cutoff point</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Round quota to</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.6 The Quota Rule and Apportionment Paradoxes

Each of the apportionment methods has both positive and negative features. Some methods are better than others but none is perfect.

Violating the Quota Rule

What is the *quota rule*?

If an apportionment method apportions to a state more than $U$ seats we say it *violates the upper quota*. If an apportionment method apportions to a state less than $L$ seats we say it *violates the lower quota*.

We will now consider how the various methods we learned in this chapter fare with respect to the quota rule.

**Hamilton’s method**

**Jefferson’s method**

**Adam’s method**

**Webster’s and the Huntington-Hill method**

**Apportionment Paradoxes**

There are three paradoxes that sometimes come up in apportionment problems:

1. **Alabama paradox:**

2. **Population paradox:**

3. **New-states paradox:**

Of the five methods, Webster’s method and the Huntington-Hill method are in some sense the best because while in theory they can violate the quota rule, in practice this happens very rarely. Of course, it would be ideal to find a method that does not violate the quota rule and does not produce any of the apportionment paradoxes.

In 1980, Michel Balinski of the State University of New York and H. Peyton Young to Johns Hopkins University were able to prove that such a method is mathematically impossible.
**Balinski and Young’s Impossibility Theorem**

If an apportionment method *does not* violate the quota rule, then apportionment paradoxes are *possible*.

If the apportionment method *does not* produce apportionment paradoxes, then violations of the quota rule are *possible*.

**Summary:**

**Homework:** Ch.4 (p.128) #1, 3, 41, 45, 47, 51-57 (odd), 60-63, 76-78