Now that we have discussed what functions are and some of their characteristics, we will explore different types of functions. Section 3.2 of the text outlines a variety of types of functions. Notice that since the following are all functions, they will all pass the Vertical Line Test.

Algebraic Functions

A function is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division and taking roots). Polynomials, power functions, and rational function are all algebraic functions.

1 Polynomials

A function $p$ is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $n$ is a nonnegative integer and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are all constants called **coefficients** of the polynomial.

If the leading coefficient $a_n \neq 0$ then the **degree** of $p(x)$ is $n$.

- A polynomial of degree 1 is called a **linear function**.
- A polynomial of degree 2 is called a **quadratic function**.
- A polynomial of degree 3 is called a **cubic function**.
- A polynomial of degree 4 is called a **quartic function**.
- A polynomial of degree 5 is called a **quintic function**.
- ...and so on...

Linear Functions

The most famous polynomial is the linear function.

$y$ is said to be a **linear function** of $x$ if the graph of the function is a line so that we can use the **slope-intercept form** of the equation of a line to write a formula for the function as

$$y = mx + b$$

where $m$ is the slope and $b$ is the $y$-intercept.

If this is the case, what do $m$ and $b$ equal in the $p(x)$ equation?
Graph the family of equations \( f(x) = x + b \) where \( b \) is an integer \( b = -2, -1, 0, 1, 2 \) on the same coordinate system.

Graph the family of equations \( f(x) = mx \) where \( m \) is an integer \( m = -2, -1, 0, 1, 2 \) on the same coordinate system.

2 Power Functions

A function of the form \( f(x) = x^a \) where \( a \) is a constant is called a power function. The power function takes a variety of forms based on the type of constant that \( a \) is.

These different forms arise when

- **2.1** \( a = n \) where \( n \) is a positive integer,
- **2.2** \( a = 1/n \) where \( n \) is a positive integer, and
- **2.3** \( a = -1 \).

We will explore these forms in the following sections.

2.1 If \( a = n \) where \( n \) is a positive integer...

\[
\begin{array}{c}
\text{\( f(x) = x^n \)}
\end{array}
\]

Notice that if \( n \) is a positive integer then the power function is really just a type of polynomial.

Using your graphing calculator, sketch a graph of the following functions:

a. \( f_1(x) = x \)

b. \( f_2(x) = x^2 \)
c. \( f_3(x) = x^3 \)  

d. \( f_4(x) = x^4 \)  

\[ f_5(x) = x^5 \]  

\[ f_6(x) = x^6 \]  

When \( n \) is odd,  

- what is the domain of \( f(x) = x^n \)?  
- what is the range of \( f(x) = x^n \)?  
- where is \( f(x) = x^n \) increasing?  
- where is \( f(x) = x^n \) decreasing?  

When \( n \) is even,  

- what is the domain of \( f(x) = x^n \)?  
- what is the range of \( f(x) = x^n \)?  
- where is \( f(x) = x^n \) increasing?  
- where is \( f(x) = x^n \) decreasing?  

2.2 If \( a = 1/n \) where \( n \) is a positive integer...  

The functions of the form \( f(x) = x^{1/n} \) are called root functions.  

Write \( f(x) = x^{1/2} \) in a different form.  

Is \( f(x) = x^{1/n} \), where \( n \) is a positive integer, a polynomial?
Using your graphing calculator as a tool, sketch a graph of the following functions:

a. \( f_2(x) = x^{1/2} \)  
b. \( f_3(x) = x^{1/3} \)  
c. \( f_4(x) = x^{1/4} \)  
d. \( f_5(x) = x^{1/5} \)

When \( n \) is odd,

- what is the domain of \( f(x) = x^{1/n} \)?
- what is the range of \( f(x) = x^{1/n} \)?
- where is \( f(x) = x^{1/n} \) increasing?
- where is \( f(x) = x^{1/n} \) decreasing?

When \( n \) is even,

- what is the domain of \( f(x) = x^{1/n} \)?
- what is the range of \( f(x) = x^{1/n} \)?
- where is \( f(x) = x^{1/n} \) increasing?
- where is \( f(x) = x^{1/n} \) decreasing?

2.2 The Semicircle Function

The \textbf{semicircle function} is of the form

\[
f(x) = \sqrt{a - x^2}
\]

where \( \sqrt{a} \) is the radius of the semicircle.
Sketch a graph of the following functions and describe them:

\[ f(x) = \sqrt{1 - x^2} \quad f(x) = \sqrt{4 - x^2} \]

Range: Range: .
Domain: Domain: .
Increasing: Increasing: .
Decreasing: Decreasing: .

2.3 If \( a = -1 \)...

In this case, \( f(x) = x^{-1} \) is the reciprocal function.

Is \( f(x) = x^{-1} \) a polynomial?

Using your graphing calculator as a tool, sketch a graph of \( f(x) = x^{-1} \) and describe the domain, range and intervals of increasing and decreasing:

Domain: 
Range: 
Increasing: 
Decreasing: 

Piecewise Functions

Piecewise functions are defined to be one of the above types of functions on one part of the \( x \)-axis and another function on a different part of the \( x \)-axis.

For example consider

\[ f(x) = \begin{cases} 
  x + 1, & \text{if } x \leq 1 \\
  x^2, & \text{if } x > 1 
\end{cases} \]

This function has the same outputs as \( g(x) = x + 1 \) for \( x \) values less than or equal to 1 (the left half of the graph) and looks like \( h(x) = x^2 \) for \( x \) greater than 1 (the right half of the graph).

Find \( f(-2) \). Find \( x \) when \( f(x) = 4 \).
Sketch a graph of $f(x)$.

Note that the absolute value function,

$$g(x) = |x|$$

is a type of piecewise defined function.

In other words,

$$g(x) = |x| = \begin{cases} 
-x, & \text{if } x < 0 \\
x, & \text{if } x \geq 0 
\end{cases}$$

Sketch the graph of $g(x) = |x|$ and describe where it is increasing and decreasing and its domain and range.

**Domain:**  
**Range:**  
**Increasing:**  
**Decreasing:**

Sketch the graphs of the following function and describe where it is increasing and decreasing and its domain and range.

$$f(x) = \begin{cases} 
x + 1, & \text{if } x < -2 \\
x^3, & \text{if } 2 \leq x \leq 2 \\
x^{1/2}, & \text{if } x > 2 
\end{cases}$$

**Range**  
**Domain:**  
**Increasing:**  
**Decreasing:**

If you have not done so already, take this time to fill out the ‘Types of Function’ worksheet.