Differentiability an Continuity

As we have already discussed, not all limits exist. Because of this it is easy to imagine how a function may not have a derivative (if that limit does not exist). When this happens we say that the function is \textit{not differentiable} at the point \( a \).

Geometrically this means that either the function does not have a tangent line at \( a \) or that it has a vertical tangent line at \( a \).

\textbf{Examples:}

Differentiability is closely related to continuity. You probably have an informal idea of what continuity is.

More formally, continuity can be defined using limits:

\begin{center}
\textbf{Continuity}

A function \( f(x) \) is \textbf{continuous} at \( x = a \) provided that
\end{center}

\textbf{Examples:}

It is important to note that \textit{continuity does not necessarily imply differentiability}. However,

\begin{center}
\textbf{Theorem 1}

If \( f(x) \) is differentiable at \( x = a \) then \( f(x) \) is continuous at \( x = a \).
\end{center}
Examples: Determine whether each of the following functions is continuous and/or differentiable.

1. \( f(x) = \begin{cases} x & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} \)

2. \( g(x) = \begin{cases} x^3 & \text{for } 0 \leq x < 1 \\ x & \text{for } 1 \leq x \leq 2 \end{cases} \)

More Rules for Differentiation

We have already discussed the Power Rule for differentiation. Two other useful rules will help us to differentiate polynomial in general.

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<th>Differentiation Rules</th>
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<td>1. Constant Multiple Rule: for ( k ) a constant, ( \frac{d}{dx}[k \cdot f(x)] = )</td>
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<td>2. Sum Rule: ( \frac{d}{dx}[f(x) + g(x)] = )</td>
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<td>3. Power Rule: ( \frac{d}{dx}[(g(x))^r] = )</td>
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Examples:

1. \( \frac{d}{dx}[3x^2] = \)

2. \( \frac{d}{dx}[x^9 + \sqrt{x}] = \)

3. \( \frac{d}{dx}[(2x + 1)^3] = \)

Proofs of Rules 1 and 2

What other results do you think that we will need to use in order to prove that these rules are true?
proof of the Constant Multiple Rule: Assume \( f(x) \) is differentiable at \( x = a \).

What we want to show is that \( k \cdot f(x) \) is also differentiable at \( x = a \) and that its derivative is \( k \cdot f'(x) \) so we should examine

\[
\frac{d}{dx}[k \cdot f(x)] =
\]

proof of the Sum Rule:

Examples:

1. Find the following.
   
   (a) \( \frac{d}{dx} \left[ \frac{3}{x} \right] \)

   (b) \( \frac{d}{dx} [x^4 - 2x] \)

   (c) \( \frac{d}{dx} [3x^7 - x^5 + 17] \)

   (d) \( \frac{d}{dx} \left[ \frac{2}{x + 1} \right] \)
(e) \( \frac{d}{dx} \left[ \sqrt{x^2 - 1} \right] \)

(f) \( \frac{d}{dx} \left[ 2x^8 - 6x^2 + x + \frac{3}{x} \right] \)

(g) \( \frac{d}{dx} \left[ 5 \sqrt{1 + x^3} \right] \)

(h) \( \frac{d}{dx} \left[ x + 3 + \sqrt{x + 3} \right] \)

(i) \( \frac{d}{dx} \left[ 12 + \frac{1}{7^3} \right] \)

(j) \( \frac{d}{dx} \left[ \left( x - \frac{1}{x} \right)^{-1} \right] \)

(k) \( \frac{d}{dx} \left[ (x^2 + 1)^2 + 3(x^2 - 1)^2 \right] \)
2. Find the equation of the tangent line to

\[ y = \frac{8}{x^2 + x + 2} \]

at \( x = 2 \).