Lagrange Multipliers and Constrained Optimization

We have seen optimization problems where we are requires to optimize an objective function where the variables were subject to a restraint equation.

Here is our problem:

Let $f(x,y)$ and $g(x,y)$ be functions of two variables. Find values of $x$ and $y$ that maximize (or minimize) the objective function $f(x,y)$ and that also satisfy the constraint equation $g(x,y) = 0$.

Normally we would like to use the function $g(x,y) = 0$ to write one variable as a function of the other. However, this is not always possible. (For instance if $g(x,y) = x^4 - 5x^3y + 7x^2y^3 + y^5 - 17 = 0$.)

The Method of Lagrange Multipliers is our way to circumvent this problem. The basic idea is to replace $f(x,y)$ by an auxiliary function $F(x,y,\lambda)$ defined as $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$.

The new variable $\lambda$ is called the Lagrange multiplier and always multiplies the constraint function $g(x,y)$.

**Theorem**

Suppose that, subject to the constraint $g(x,y) = 0$, the function $f(x,y)$ has a relative minimum or maximum at $(a,b)$. Then there is a value $\lambda$—say $\lambda = c$—such that the partial derivatives of $F(x,y,\lambda)$ all equal zero at $(x,y,\lambda) = (a,b,c)$.

What is this theorem telling us?
Examples

1. Maximize $36 - x^2 - y^2$ subject to the constraint $x + 7y - 25 = 0$.

2. Minimize $42x + 28y$ subject to the constraint $600 - xy = 0$ where $x$ and $y$ are restricted to positive values.
Double Integrals

So far we have only been discussing derivatives of functions of several variable.

Our generalization of the definite integral for a function of one variable is the double integral over the region $R$ denoted

$$\int \int _R f(x,y) \, dx \, dy.$$ 

What is $R$?

What does the double integral mean?

Examples

1. $\int \int _{[1,2] \times [3,4]} (y - x) \, dy \, dx$
2. \[ \int_{0}^{1} \left( \int_{0}^{4} x \sqrt{y} + y \ dy \right) \ dx \]

3. \[ \int_{0}^{1} \left( \int_{\sqrt{x}}^{x+1} 2xy \ dy \right) \ dx \]