Types of Functions

Now that we have discussed what functions are and some of their characteristics, we will explore different types of functions. Section 0.2 of the text outlines a variety of types of functions. Notice that since the following are all functions, they will all pass the Vertical Line Test.

Algebraic Functions

A function is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division and taking roots). Polynomials, power functions, and rational function are all algebraic functions.

1 Polynomials

A function \( p \) is a polynomial if

where \( n \) is a nonnegative integer and \( a_0, a_1, a_2, \ldots, a_{n-1}, a_n \) are all constants called coefficients of the polynomial.

If the leading coefficient \( a_n \neq 0 \) then the degree of \( p(x) \) is \( n \).

A polynomial of degree 1 is called a linear function.

A polynomial of degree 2 is called a quadratic function.

A polynomial of degree 3 is called a cubic function.

A polynomial of degree 4 is called a quartic function.

A polynomial of degree 5 is called a quintic function.

...and so on...

Linear Functions

The most famous polynomial is the linear function.

\( y \) is said to be a linear function of \( x \) if the graph of the function is a line so that we can use the slope-intercept form of the equation of a line to write a formula for the function as

where \( m \) is the slope and \( b \) is the \( y \)-intercept.

If this is the case, what do \( m \) and \( b \) equal in the \( p(x) \) equation?
Graph the family of equations $f(x) = x + b$ where $b$ is an integer $b = -2, -1, 0, 1, 2$ on the same coordinate system.

Graph the family of equations $f(x) = mx$ where $m$ is an integer $m = -2, -1, 0, 1, 2$ on the same coordinate system.

Find and graph the equation for a line that passes through $(2, -1)$ and $(3, 5)$.

Quadratic Function

A quadratic function is a function of the form

where $a \neq 0, b, c$ are constants. The shape of this graph is a parabola and the vertex of the parabola is at the point:
2 Power Functions

A function of the form \( f(x) = x^a \) where \( a \) is a positive integer is called a **power function**. Notice that if \( n \) is a positive integer then the power function is really just a type of **polynomial**.

Using your graphing calculator, sketch a graph of the following functions:

a. \( f_1(x) = x \)  

b. \( f_2(x) = x^2 \)

c. \( f_3(x) = x^3 \)  
d. \( f_4(x) = x^4 \)

e. \( f_5(x) = x^5 \)  
f. \( f_6(x) = x^6 \)

When \( n \) is odd,

- what is the domain of \( f(x) = x^n \)?
- what is the range of \( f(x) = x^n \)?
- where is \( f(x) = x^n \) increasing?
- where is \( f(x) = x^n \) decreasing?
- How do you know that these characteristics will hold for *every* odd \( n \)?

When \( n \) is even,

- what is the domain of \( f(x) = x^n \)?
- what is the range of \( f(x) = x^n \)?
- where is \( f(x) = x^n \) increasing?
- where is \( f(x) = x^n \) decreasing?
- How do you know that these characteristics will hold for *every* even \( n \)?
3 Rational Functions

Let’s look at \( f(x) = x^{-1} \) is the **reciprocal function**.

Is \( f(x) = x^{-1} \) a polynomial?

Using your graphing calculator as a tool, sketch a graph of \( f(x) = x^{-1} \) and describe the domain, range and intervals of increasing and decreasing:

**Domain:**
**Range:**
**Increasing:**
**Decreasing:**

The reciprocal function is a type of rational function. A **rational function** is a ratio of two polynomials, \( p(x) \) and \( q(x) \):

\[
f(x) = \frac{p(x)}{q(x)}
\]

Which of the previously mentioned functions is a rational function?

What happens when you evaluate this function where \( q(x) = 0 \)?

Consider \( f(x) = \frac{x-1}{x^2-4} \).

- Identify \( p(x) \) and \( q(x) \).

- Using your graphing calculator, sketch a graph of \( f(x) \).

- What happens at \( x = 2 \) and \( x = -2 \)? Why?
Piecewise Functions

Piecewise functions are defined to be one of the above types of functions on one part of the $x$-axis and another function on a different part of the $x$-axis.

For example consider

$$f(x) = \begin{cases} 
  x + 1, & \text{if } x \leq 1 \\
  x^2, & \text{if } x > 1
\end{cases}$$

This function has the same outputs as $g(x) = x + 1$ for $x$ values less than or equal to 1 (the left half of the graph) and looks like $h(x) = x^2$ for $x$ greater than 1 (the right half of the graph).

Sketch a graph of $f(x)$.

Find $f(-2)$.

Find $x$ when $f(x) = 4$.

Note that the absolute value function is a type of piecewise defined function.
In other words,

$$g(x) = |x| = \begin{cases} 
  -x, & \text{if } x < 0 \\
  x, & \text{if } x \geq 0
\end{cases}$$

Sketch the graph of $g(x) = |x|$ and describe where it is increasing and decreasing and its domain and range.

Domain:
Range:
Increasing:
Decreasing:
Algebra of Functions

Many functions can be viewed as combinations of other functions. For instance,

\[ P(x) = R(x) - C(x). \]

The operations on functions are similar to those of real numbers, with one exception. As with real numbers, with two function \( f(x) \) and \( g(x) \), we can

- Add
- Subtract
- Multiply and
- Divide

them to get a new function.

**Examples:** Let \( f(x) = 2x + 4 \), \( g(x) = -x + 1 \), \( h(x) = \frac{3}{x} \), and \( k(x) = \frac{4}{x-2} \). Simplify the following operations.

1. \( f(x) + g(x) \)
2. \( f(x) - g(x) \)
3. \( g(x)h(x) \)
4. \( \frac{f(x)}{g(x)} \)
5. \( \frac{h(x)}{k(x)} \)
6. \( h(x) + k(x) \)

However, there is one operation that functions have that real numbers don’t:

The **composition** of \( f(x) \) and \( g(x) \) is _______________.

Will \( f(g(x)) = g(f(x)) \) for each \( f \) and \( g \)? Why or why not?
Examples: For $f, g, h$ and $k$ as above, simplify:

6. $f(g(x))$

7. $g(f(x))$

8. $h(k(x))$