Limits and Derivatives

Recall that when we were calculating the derivative informally, in the last step we let ‘$h$ approach 0’ in the quotient, what we were really doing was taking a limit. The concept of a limit is integral in terms of computing the derivative of a function.

The Derivative Problem

Consider the function

$$f(x) = 3x^2 - 1.$$ 

Sketch a graph of this function, labeling the axes.

How do you find the average rate of change of this function if you know that at $a_1$ it is at position $f(a_1)$ and at time $a_2$ it is at position $f(a_2)$?

The difficulty with finding the rate of change at $x = 3$, is that we are dealing with a single point on the $x$-axis as opposed to an interval. However, we can approximate the desired quantity by computing the average velocity over the brief time interval $a_1 = 3.0$ and $a_2 = 3.1$.

Compute the average rate of change of $f$ between these two points.

Again, what you calculated is the average rate of change on an interval close to $x = 3$ but not the exact rate of change at $x = 3$. To get an even more accurate value, complete the following table.
<table>
<thead>
<tr>
<th>Input Interval</th>
<th>Average ROC</th>
<th>Input Interval</th>
<th>Average ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \leq x \leq 4$</td>
<td></td>
<td>$2 \leq x \leq 3$</td>
<td></td>
</tr>
<tr>
<td>$3 \leq x \leq 3.1$</td>
<td></td>
<td>$2.9 \leq x \leq 3$</td>
<td></td>
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<tr>
<td>$3 \leq x \leq 3.05$</td>
<td></td>
<td>$2.95 \leq x \leq 3$</td>
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<tr>
<td>$3 \leq x \leq 3.01$</td>
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<td>$2.99 \leq x \leq 3$</td>
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<tr>
<td>$3 \leq x \leq 3.001$</td>
<td></td>
<td>$2.999 \leq x \leq 3$</td>
<td></td>
</tr>
</tbody>
</table>

What do you notice about your results?

Refer back to the graph of the $f(x) = 3x^2 - 1$. How does the chart that you completed relate to this graph? (You may want to mark the function at the intervals that you are discussing.)

As it turns out, the numbers that you were calculating were the *slopes of the secant lines that pass through the graph and the indicated time values.*

As your intervals get smaller and smaller (the endpoints get closer to 3), you are getting closer and closer to finding the slope of the line tangent to (touching but not crossing through) the function $f(x)$ at $x = 3$. In other words, you are getting closer and closer to finding the derivative at $x = 3$.

What would you estimate $f'(3)$ is, based on your chart?

Notice that this confirms the derivative that we calculated last time.

The rest of this chapter will be discussing techniques for more accurately arriving at this value called the *derivative.* The main tool that we will be discussing in what follow is the concept of the *limit.*
Limits

**LIMIT OF A FUNCTION**

Let \( g(x) \) be a function and \( a \) a number. We say that the number \( L \) is the **limit of \( g(x) \) as \( x \) approaches \( a \)** provided that \( g(x) \) can be made arbitrarily close to \( L \) for \( x \) sufficiently close (but not equal) to \( a \). In this case we write

\[
\lim_{x \to a} g(x) = L
\]

What does this mean?

**EXAMPLES:**

Determine if \( \lim_{x \to 3} g(x) \) exists for each of the following functions:

Computing a limit is not usually too bad if you have a chart or a graph, but just given the algebraic expression of the function \( g \) things can be a bit trickier. The following rules are helpful in such a case.

**LIMIT THEOREMS**

If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist then

(I) If \( k \) is a constant, then \( \lim_{x \to a} k \cdot f(x) = k \cdot \lim_{x \to a} f(x) \)

(II) If \( r \) is a positive constant and \( [f(x)]^r \) is defined for \( x \neq a \) then \( \lim_{x \to a} [f(x)]^r = [\lim_{x \to a} f(x)]^r \)

(III) \( \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \)

(IV) \( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \)

(V) \( \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \)

(VI) If \( \lim_{x \to a} g(x) \neq 0 \) then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \)

**EXAMPLES:** Use these Theorems to evaluate the following limits.

(a) \( \lim_{x \to 2} x^3 \)
(b) \( \lim_{x \to 2} 4x^3 \)

(c) \( \lim_{x \to 2} \sqrt{4x^3 - 7} \)

**Limit Theorems [continued]**

(VII) Let \( p(x) \) be a polynomial function and \( a \) a number then \( \lim_{x \to a} p(x) = p(a) \).

(VIII) Let \( r(x) = \frac{p(x)}{q(x)} \) be a rational function and \( a \) a number such that \( q(a) \neq 0 \) then \( \lim_{x \to a} r(x) = r(a) \).

**But what happens if the rational function satisfies** \( q(a) = 0 \)? There are two main tricks:

1. **Factor and simplify.**
   
   **Example:** Compute \( \lim_{x \to 4} \frac{x^2 - 16}{x - 4} \)

2. **Rationalize the numerator.**
   
   **Example:** Compute \( \lim_{x \to 0} \frac{\sqrt{x + 25} - 5}{x} \)
We can also use limits in order to determine the end behavior a function. Instead of letting $x$ approach some number $a$, we look at the function as $x$ approached $\pm\infty$.

**Examples:** Compute the following limits.

1. $\lim_{x \to \infty} \frac{1}{x^3 + 5}$
2. $\lim_{x \to \infty} \frac{3x + 1}{2x - 7}$

**Derivatives**

Now that we know how to compute limits, this brings up to our formal definition of the derivative.

**Derivative as a limit**

$$f'(x) =$$

We can use this definition to properly compute derivatives (i.e. slopes of curves).

**Examples:** Compute the derivatives of the following functions.

1. $f(x) = \frac{1}{x}$
2. $f(x) = \sqrt{x}$