Applications of Exponential Functions

Differentiate \( y = Ce^{kt} \).

What do you notice?

Exponential Growth and Decay

When at every instant, the rate of increase of a quantity is proportional to the quantity at that instant we say the quantity is exhibiting exponential growth. The classic example is the growth of a bacteria culture in a lab–The number of bacteria present is proportional to its growth rate since the more bacteria that exists at a single instant, the greater the possibility of division.

We can model this phenomena with a simple differential equation:

Let \( P(t) \) denote the number of bacteria in a certain culture at time \( t \).

What is \( P'(t) \)?

The exponential growth of the bacteria is then modeled by the equation

\[
P'(t) = kP(t)
\]

for some constant \( k \).

What do you notice?

When \( k \) is a negative number we call this exponential decay.

One of the more interesting types of exponential decay models is Radiocarbon Dating. See the number 2 and 3 below for more details.
Examples

1. A certain bacteria culture grows at a rate proportional to its size. At time \( t = 0 \) approximately 20,000 bacteria are present. In 5 hours there are 400,000 bacteria. Determine a function that expresses the size of the culture as a function of time, measured in hours.

   (a) Find a mathematical model for the population level.

   (b) How large will the population be after 10 days?

   (c) How fast will the population be growing after 10 days?

2. A population is growing exponentially with growth constant .04. In how many years will the population double?

3. Radioactive carbon 14 has a half-life is about 5730 years. Find its decay constant.
All living vegetation and most animals contain carbon 14 and carbon 12 in the same proportion as the atmosphere. When an organism dies its stops replacing its carbon. The carbon 14 begins to decrease through radioactive decay but the carbon 12 in the dead organism remains constant.

4. A parchment fragment was discovered that had about 80% of the carbon 14 level found today in living matter. Estimate the age of the parchment.

**Compound Interest**

Recall that if a principal amount \( P \) is compounded \( m \) times per year at a rate of interest \( r \) for \( t \) years, the compound amount \( A \) (the balance at the end of time \( t \)) is given by

\[
A = P \left( 1 + \frac{r}{m} \right)^{mt}.
\]

Suppose that $1000 is invested at 6% interest for 1 year.
If the interest is *compounded annually*, how much would be in the account at the end of the year?

If the interest is *compounded quarterly*, how much would be in the account at the end of the year?

If the interest is *compounded daily*, how much would be in the account at the end of the year?

Would the account balance be much more if the interest were compounded every hour?

…every minute?

To see why this is let’s examine the compound amount formula.
Observe that
\[ A = P \left(1 + \frac{r}{m}\right)^{mt} = P \left(1 + \frac{r}{m}\right)^{(m/r)rt}. \]

Let \( h = r/m \) and we have

What happens to \( h \) as \( m \) gets large?

Thus if we want to compute the amount when the interest is compounded continuously we must compute:

Recall that we defined \( e \) as the number so that
\[ \lim_{h \to 0} \frac{e^h - 1}{h} = 1. \]

From this definition, we have:

<table>
<thead>
<tr>
<th><strong>Definition</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>The number ( e ) is defined so to be the number so that</td>
</tr>
<tr>
<td>[ \lim_{h \to 0} (1 + h)^{1/h} = e. ]</td>
</tr>
</tbody>
</table>

What does this tell us about the compound interest formula above?
Examples

1. Suppose $1000 is invested at 5% interest compounded continuously.
   (a) Give the formula for the compound amount after $t$ years $A(t)$.
   
   (b) How much will be in the account after 1 year?

   (c) How much will be in the account after 6 years?

   (d) After 6 years at what rate with $A(t)$ be growing?

   (e) How long will it take in order for the initial investment to double?

2. Find the present value of $5000 to be received in 2 years if the money can be invested at 12% compounded continuously.
3. Pablo Picasso’s *The Dream* was purchased in 1941 for a war-distressed price of $7000. The painting was sold in 1997 for $48.4 million, the second highest price ever paid for a Picasso painting at auction. What rate of interest compounded continuously did this investment earn?