Area in the xy-Plane

Recall that in the previous Section 6.3 we have found the area under a curve by approximating using rectangles. The more rectangles, the better the approximation. We saw that if we were to take infinitely many rectangles, we would no longer be approximating the area, we would be computing it exactly.

**Definition**

The area $A$ of a region $S$ that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{\Delta x \to 0} \left[ f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x \right]$$

Furthermore, the Fundamental Theorem of Calculus has shown us that this infinite limit of sums of areas is related to the definite integral. Specifically,

**Fundamental Theorem of Calculus**

Let $f$ be a continuous function on the interval $[a, b]$ with an antiderivative $F$. Then, the Riemann sums approach the definite integral of $f$ on $[a, b]$ as the number of subintervals increases indefinitely. That is,

$$A = \lim_{\Delta x \to 0} \left[ f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x \right] =$$

Notice though that if the function is below the $x$-axis, this will be the negative area:

What happens if the function is both above and below the $x$-axis?
Examples:

1. Find the area between the curve $f(x) = x^2 - x$ and the $x$-axis on the interval from $[0, 2]$.

   What if we wanted to find the area between two curves?

2. Find the area between the curves $y = x^2 + 1$ and $y = -x^2 - 1$ from $x = -1$ to $x = 1$.

3. Find the area of the region bounded by $y = x^2$, $y = x^2 - 4x + 4$, $x = 0$, and $x = 3$. 

4. Find the area of the region bounded by $y = x^2 - 2x$, $y = -e^x$, $x = -1$, and $x = 2$.

Applications of the Definite Integral

Average Value of a Function

We can use the definite integral to find the average value of a function. How would you do to calculate the average value of a collection of numbers $y_1, y_2, ..., y_n$?

Similarly we can approximate the average value of a function by finding the average value of $f$ at points say $x_1, x_2, ..., x_n$:

How can we turn this into something representing the definition of the definite integral?
**Average Value of a Function**

The **average value of a function** $f(x)$ over the interval $a \leq x \leq b$ is defined as the quantity

\[
\frac{1}{b-a} \int_a^b f(x) \, dx.
\]

**Example:** Compute the average value of $f(x) = \sqrt{x}$ over the interval $0 \leq x \leq 9$.

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**Consumers’ Surplus**

The **consumers’ surplus** calculates how much a consumer benefits from items being sold at a particular price. We assume that $A$ units are being sold at price $B$ what we want to calculate is how much the consumers would need to pay if less units were being sold. For instance, consider the demand curve $p = f(x)$.

\[
\text{where the quantity demand is } A \text{ and the price is } B = f(A).
\]
Example: Find the consumers’ surplus for the demand curve \( p = 50 - .06x^2 \) at the sales level \( x = 20 \).

Future Value of an Income Stream

Such a problem is best demonstrated through an example...

Example: Suppose that money is deposited daily into a savings account so that \$1000 is deposited each year. The account pays 6% interest compounded continuously. Approximate the amount of money in the account at the end of five years.

**Future Value of an Income Stream**

The future value of a continuous income stream of \( K \) dollars per year for \( N \) years at an interest rate \( R \) compounded continuously is

\[
\text{Future Value} = Ke^{Rt}
\]
Volume of the Solid of Revolution

When a region is revolved about the \( x \)-axis, it sweeps out a solid. Riemann sums can be used to derive a formula for this solid of revolution.

Sketch a picture of a curve being rotated about the \( x \)-axis.

How would you approximate its volume?

\[\text{Volume of the Solid of Revolution}\]

The volume of the \textbf{solid of revolution} obtained from revolving the region below the graph of \( y = g(x) \) from \( x = a \) to \( x = b \) about the \( x \)-axis is

\[
\text{Volume} = \int_a^b \pi [g(x)]^2 \, dx
\]

Example: Find the volume of a solid of revolution we obtain by revolving the region defined \( y = e^{kx} \) from \( x = 0 \) to \( x = 1 \) about the \( x \)-axis.