Newton’s Method

We already know how to find the roots of many polynomial functions.

For instance, if the function is factorable, we can factor it and set it equal to zero.

If the function is not obviously factorable, and is degree 2, 3, or 4, we can use a quadratic, cubic or quartic equation to find it’s roots.

However, no such equations exist for polynomials of degree higher than 4. (This is actually a proven result that motivated the creation of a whole branch of mathematics that has led to a boundless research since the early 19th century.)

**Newton’s Method** allows up to approximate the roots of functions that we are not able to identify. This is actually the approximation method that your calculator uses to identify such roots.

**Example**

To explore how Newton’s Method works, consider the function

\[ f(x) = x^5 - x - 1. \]

a. Your calculator approximates the following roots:

As a first approximation...

b. The geometry can be seen as follows:

(1) Sketch a graph of \( f(x) = x^5 - x - 1. \)
(2) Label the point at which \( x_1 = 1 \) (i.e. the point \((1, f(1))\)) on the graph.
(3) Sketch the tangent line \( L_1 \) to the curve \( f(x) \) at the point \((1, f(1))\).
(4) Label the \( x \)-intercepts of \( L_1 \) and call it \( x_2 \).
c. What do you notice about $x_2$?

Because $L_1$ is the tangent line to $f(x)$ at $x_1 = 1$ (i.e. the point $(1, f(1))$), we can easily find the equation for $L_1$.

d. Find the equation for $L_1$.

e. Set $y = 0$ for your equation for $L_1$ and solve for $x_2$. What is the significance of this value that you found?

As a second approximation...

f. Now, find the equation for the tangent line $L_2$ to $f(x)$ at $(x_2, f(x_2))$.

Solve this equation for $x_3$ for when $y = 0$.

Explain what the value of $x_3$ represents. (You may want to use the graph you sketched above.)
What would you imagine would happen if we continued on this process?

If we take \( \lim_{n \to \infty} x_n \) we will get the root that we desired.

Exercises

1. Use Newton’s Method with the special initial approximation \( x_1 \) to find \( x_3 \), the third approximation to the root of the given equation.

   (a) \( x^3 + 2x - 4 = 0, \ x_1 = 1 \)

   (b) \( x^5 + 2 = 0, \ x_1 = -1 \)
2. Use Newton’s method to approximate \( \sqrt[5]{20} \) correct to eight decimal places.

3. Use Newton’s method to find all solutions to \((x - 2)^2 = \ln x\) correct to six decimal places.