U-Substitution

Warm-Up

Evaluate the following definite integrals:

1. \( \int_{0}^{1} x^{4/5} \, dx \)

2. \( \int_{-5}^{5} e \, dx \)

3. \( \int_{-1}^{1} t(1 - t)^2 \, dt \)

4. \( \int_{1}^{2} \frac{x^3 + 3x^6}{x^4} \, dv \)

The Indefinite Integral

The information that we have at this point is sufficient to compute the previous integrals. However, it is not too difficult to imagine a function that we would be unable to integrate at this point.

For instance,

\[
\int 2x\sqrt{1 + x^2} \, dx.
\]

We cannot algebraically simplify \( 2x\sqrt{1 + x^2} \) like in the last two Warm-Up exercises. So instead, we will introduce a new variable.

In this particular example, our variable that we will want to introduce is

\[ u = 1 + x^2. \]

Now, find the differential of this new variable \( u = 1 + x^2 \).
Referring back to our integral, notice that
\[ \int 2x\sqrt{1 + x^2} \, dx = \int \sqrt{1 + x^2} \cdot 2x \, dx = \int \sqrt{u} \, du. \]

Now, integrate \( \int \sqrt{u} \, du. \) (Note: Don't forget the ‘+C’ since this is an indefinite integral!)

Finally, substitute \( u = 1 + x^2 \) back in for \( u. \)

That is your integral!
Check to make sure that your integration is correct.

Notice that this technique (often referred to as ‘u-substitution’) can be thought of as the integration equivalent to the Chain Rule.

**THE SUBSTITUTION RULE**

If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) and continuous on \( I, \) then
\[
\int f(g(x))g'(x) \, dx = \int f(u) \, du
\]

**Examples**
Evaluate the following indefinite integrals:

1. \( \int x \ln(x^2) \, dx \)
2. \[ \int \frac{x}{(x^2+1)^2} \, dx \]

3. \[ \int e^x(e^x+1)^5 \, dx \]

The Definite Integral

Since we have already computed \( \int 2x\sqrt{1+x^2} \, dx \), we know \( F(x) \) (an antiderivative of \( 2x\sqrt{1+x^2} \)).

\[ \int_{-1}^{2} 2x\sqrt{1+x^2} \, dx. \]

Since we have already computed \( \int 2x\sqrt{1+x^2} \, dx \), we know \( F(x) \) (an antiderivative of \( 2x\sqrt{1+x^2} \)).

We have two options for how we want to compute the definite integral...

**Option 1:**
We can use the *Evaluation Theorem* (or Part 2 of the Fundamental Theorem of Calculus) in order to figure out how to apply u-substitution to evaluate definite integrals.
Option 2:
Use the Substitution Rule for Definite Integrals.

The Substitution Rule for Definite Integrals

If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of
$u = g(x)$, then

$$
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
$$

Examples

Evaluate the following definite integrals:

1. $\int_0^1 \ln(3t) \, dt$

2. $\int_0^1 \sqrt{1 + 7x} \, dx$
3. \( \int_{4}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \)

4. \( \int_{0}^{1} \frac{e^{z} + 1}{e^{z} + z} \, dz \)

5. \( \int_{e^{4}}^{e^{4}} \frac{dx}{x \ln x} \)