Integration Techniques

In Calculus I (Math 1271) we learned about two major operations on functions: differentiation and integration. The derivative of a function is the slope of its tangent line while the integral of the function provides us with the area under its graph.

What do we use to define this area under the curve (the integral)?

In Calculus I, we also learned that differentiation and integration are inverses of one another. This is established by the Fundamental Theorem of Calculus:

This provides us with a way of finding the integral of many functions without computing their Riemann sums. Instead we can use antiderivatives. In doing so we can use a variety of the rules for differentiation in order to derive rules for integration. For the first part of this semester we will investigate several rules for integration that will help us to integrate tricky functions whose antiderivatives would be difficult to guess off-hand.

5.5 Substitution Rule

In Calculus I, we already learned one rule for integrating difficult functions. The Substitution Rule is derived from the Chain Rule for differentiation.

**Substitution Rule**

If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x)) \, g'(x) \, dx =
\]

Examples

Evaluate the following integrals:
1. $\int x^2 \sqrt{x^3 + 1} \, dx$

2. $\int xe^{x^2} \, dx$

3. $\int 3x \sqrt{3x + 3} \, dx$

4. $\int_0^1 e^x (e^x + 1)^5 \, dx$

### 7.1 Integration by Parts

The goal of this section is to be able to integrate yet more complicated functions. For example, we currently have no means to evaluate

$$\int x \ln x \, dx.$$  

We have discussed that our Substitution Rule corresponds to the Chain Rule for differentiation. We will also have a rule that corresponds to the Product Rule for differentiation.
Recall that the *Product Rule* states that if \( f \) and \( g \) are differentiable functions, then

\[
\frac{d}{dx} \left[ f(x)g(x) \right] =
\]

Integrating both sides, this becomes

\[
\int u \, dv =
\]

Splitting up the integral, we see

\[
\int u \, dv =
\]

We can rearrange this equation and we get

\[
\int u \, dv =
\]

This provides us this the formula for Integration by Parts,

**Integration by Parts**

Let \( u = f(x) \) and \( v = g(x) \). Then the differentials \( du = f'(x) \, dx \) and \( dv = g'(x) \, dx \)

\[
\int u \, dv =
\]

This means that in order to find an antiderivative in this form we must:

**Examples**

1. Evaluate the following indefinite integrals.
   
   (a) \( \int xe^x \, dx \)
(b) \( \int x \ln x \, dx \)

(c) \( \int x \sin x \, dx \)

(d) \( \int p^5 \ln p \, dp \)

We can adopt a similar strategy for definite integrals.

\[
\int_a^b f(x)g'(x) \, dx = f(x)g(x) \bigg|_a^b - \int_a^b g(x)f'(x) \, dx
\]
2. Evaluate the following definite integrals.

(a) \( \int_1^2 \frac{\ln x}{x^2} \, dx \)

(b) \( \int_0^1 (x^2 + 1)e^{-x} \, dx \)

(c) \( \int_0^1 \frac{x}{e^{2x}} \, dx \)
Two Important Cases of IBP:

(Example 2) Natural Logarithms:
Oddly enough before this point, we did not have an antiderivative for ln\(x\). We can use Integration by Parts to evaluate

\[
\int \ln x \, dx.
\]

(Example 6) The Reduction Formula:
We can use Integration by Parts to find a general form for the integral

\[
\int \sin^n x \, dx
\]

where \(n \geq 2\) is an integer.