How to show that infinitely many mathematical statements are true:

To prove a statement of the form

\[ \forall n \in \mathbb{N} \ P(n) \]

there are two main methods

(1) Suppose we are given an arbitrary \( n \in \mathbb{N} \) and show that \( P(n) \) is true.

(2) Use Mathematical Induction.

### Mathematical Induction

**Principle of Mathematical Induction**

Let \( P(n) \) be a mathematical statement. If

(a) \( P(1) \) is true, and

(b) for every \( n \in \mathbb{N} \), \( P(n) \Rightarrow P(n+1) \) is true

Then \( P(n) \) is true for every \( n \in \mathbb{N} \).

Why does this work?

How do we use it?
Examples

1. Use mathematical induction to show that for every \( n \in \mathbb{N} \):

\[
n! \leq n^n.
\]

2. Use mathematical induction to show that for every \( n \in \mathbb{N} \):

\[
1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.
\]
3. Use mathematical induction to show that for every $n \in \mathbb{N}$:

$$2 + 4 + \ldots + 2n = n^2 + n.$$ 

4. Use mathematical induction to show that for every $n \in \mathbb{N}$:

$$8 \mid 3^{2n} - 1.$$
Strong of Mathematical Induction

Let $P(n)$ be a mathematical statement. If

(a) $P(1)$ is true, and

(b) for every $n \in \mathbb{N}$, if each of $P(1), \ldots, P(n)$ is true then $P(n + 1)$ is true

Then $P(n)$ is true for every $n \in \mathbb{N}$.

*This is actually (surprisingly) equivalent to the original version of mathematical induction!*

Examples

1. Suppose $a$ and $b$ are positive integers. Let

$$c_n = \sqrt{b}[(a + \sqrt{b})^n - (a - \sqrt{b})^n].$$

   (a) Calculate $c_0, c_1, c_2$ and $c_3$.

   (b) Show that $c_n \in \mathbb{N}$ for $n \in \mathbb{N}$.
2. Let $A$ be a subset of $\mathbb{N}$ that satisfies the following conditions:

(i) $1 \in A$,
(ii) for every $n \in \mathbb{N}$, $n \in A \implies n + 3 \in A$ and
(iii) for every $n \in \mathbb{N}$, $n \in A \implies n + 4 \in A$

(a) Find two different sets $A$ that satisfy conditions (i) and (ii).

(b) Use Strong Inductions to show that every integer greater than 6 is in $A$. 