Applications of Extrema (4.4)

<table>
<thead>
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<th>DEFINITIONS</th>
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<td>Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a force field acting on a very small object with mass $m$ which moves along a path $x: [a, b] \to \mathbb{R}^3$.</td>
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<td>$\frac{1}{2} |x'(t)|^2$ is called the ____________</td>
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<td>The force acting on the object can be expressed as</td>
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<td>$F(x(t)) = \vdots$ (Newton’s second law of motion)</td>
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<td>As it turns out, most force fields are actually gradient vector fields, so there is some potential function $V: \mathbb{R}^3 \to \mathbb{R}$ such that $F = -\nabla V$. (Notice the negative!) When we evaluate the function $V$ along the path of the object,</td>
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<td>is called the <strong>potential energy of the object</strong>.</td>
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<td>The <strong>total energy</strong> of the object is the sum of the two energies above, is written as</td>
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<td>, and is <strong>constant</strong> for all times $t$.</td>
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1. Let $F = (-2x - 2y - 1, -2x - 6y - 2)$ be a vector field representing a force in the $x, y$-plane.
   
   (a) Show that $F$ is conservative and has the potential function $V(x, y) = x^2 + 2xy + 3y^2 + x + 2y$.
   
   (b) Find a second potential function for $F$. (Hint: “...+2016”.)
1. (Continued) Still... \( \mathbf{F} = (-2x - 2y - 1, -2x - 6y - 2) \), and \( V(x, y) = x^2 + 2xy + 3y^2 + x + 2y \).

(c) What are the critical points of \( V \)? In this physical application, they are also called \textbf{equilibrium points of} \( \mathbf{F} \), because at these points \( \mathbf{F} = 0 \) (so it is not pushing at all).

(d) Which equilibrium points are \textbf{stable}, meaning that \( \mathbf{F} \) pushes particles towards them.

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Double and Iterated Integrals (5.1-2)

If your Calc II is shaky, you may want to review some integration techniques!
Sketch the region of integration and then evaluate the integrals. Convert double integrals to iterated integrals.

2. \( \int_0^2 \int_1^3 (x^2 + y) \, dy \, dx \)

3. \( \int_1^2 \int_0^1 (e^{x+y} + x^2 + \ln y) \, dx \, dy \)
4. $\int_0^2 \int_0^{x^2} y \, dy \, dx$

5. $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} 3 \, dx \, dy$  
   Hint: Trig substitution gives $\int \sqrt{1-x^2} \, dx = \frac{1}{2}(x\sqrt{1-x^2} + \arcsin x)$.

6. $\int \int_D e^{x^2} \, dA$, where $D$ is the triangle with vertices $(0,0), (1,0), (1,1)$. 
7. Find the volume under the plane $4x + 2y - z + 25 = 0$ and over the region in the $xy$-plane bounded by $y = x^2 - 10$ and $y = 31 - (x - 1)^2$.

Changing the Order of Integration (5.3)

8. Sketch the region of integration, reverse the order of integration, then evaluate both integrals.

$$
\int_0^2 \int_0^{4-y^2} x \, dx \, dy
$$
9. Sketch the region of integration, reverse the order of integration, then rewrite as a single iterated integral and evaluate.

\[ \int_0^1 \int_0^x \sin x \, dy \, dx + \int_1^2 \int_0^{2-x} \sin x \, dy \, dx \]

10. **Why we need to change the order:** Attempt to integrate in the given order.

Then reverse the order of integration and evaluate

\[ \int_0^2 \int_{y^2}^4 y \cos (x^2) \, dx \, dy \]
11. **Constructing iterated integrals from smaller integrals:**

The integral \( \int_0^1 \sqrt{x} \, dx \) measures the of the region below the function \( f(x) = \sqrt{x} \), but so does the double integral \( \int_0^1 \int_0^\sqrt{x} 1 \, dy \, dx \).

Similarly, we can write the volume of the region above the disc and below the plane \( z = 4 - x - y \) as the double integral

\[
\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4 - x - y \, dy \, dx,
\]

and we can write it as a triple integral

\[
\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x-y} 1 \, dz \, dy \, dx.
\]

Evaluate the double and triple integrals in the last sentence to confirm they are equal.

12. \( \int_0^1 \int_1^2 \int_{z}^{y+\pi} z \, dx \, dz \, dy \)