What is a **parametrized surface** and how does it differ from a parametrized **curve**?

**Smooth**

A parametrized surface $S = \mathbf{X}(D)$ is **smooth** at $\mathbf{X}(s_0, t_0)$ if the map $\mathbf{X}$ is of class $C^1$ in a neighborhood of $(s_0, t_0)$ and if the vector

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**Surface Area of $S$**

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1. Consider the parametrized surface $X(s, t) = (s^2 \cos t, s^2 \sin t, s)$ for $-3 \leq s \leq 3$, $0 \leq t \leq 2\pi$.

   (a) Find the normal vector at $(s, t) = (-1, 0)$. Is $X$ smooth at $(-1, 0)$?

   (b) Determine the tangent plane at the point $(1, 0, -1)$.

   (c) Find an equation for the image of $X$ in the form $F(x, y, z) = 0$.

2. Given the sphere of radius 2 centered at $(2, -1, 0)$ find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

   (a) by constructing the sphere as the graph of the function $f(x, y) = \sqrt{4 - (x - 2)^2 - (y + 1)^2}$.

   (b) by constructing the sphere as level surface of the function $F(x, y, z) = (x - 2)^2 - (y + 1)^2 + z^2$.

   (c) by constructing the sphere as the surface parametrized by $X(s, t) = (2 \sin s \cos t + 2, 2 \sin s \sin t - 1, 2 \cos s)$. 
3. Represent the surface given by lower hemisphere \(x^2 + y^2 + z^2 = 9\) including the equatorial circle as a piecewise smooth parametrized surface.

4. Recall that the torus is parametrized by

\[
\begin{align*}
    x &= (a + b \cos t) \cos s \\
    y &= (a + b \cos t) \sin s \\
    z &= b \sin t
\end{align*}
\]

for \(0 \leq s, t \leq 2\pi\) and \(a > b > 0\).

Find the surface area of the torus.
5. Find the area of the surface cut from the paraboloid \( z = 2x^2 + 2y^2 \) by the planes \( z = 2 \) and \( z = 8 \).

Surface Integrals (7.2)

**Surface Integrals**

Let \( \mathbf{X} : D \to \mathbb{R}^3 \) be a smooth parametrized surface where \( D \subset \mathbb{R}^2 \) is a bounded region. Let \( f \) be a continuous function whose domain includes \( S = \mathbf{X}(D) \). Then the **scalar surface integral** of \( f \) along \( \mathbf{X} \) is

\[
\int \int_{\mathbf{X}} f \, dS =
\]

Let \( \mathbf{F}(x,y,z) \) be a continuous vector field whose domain includes \( S = \mathbf{X}(D) \). Then the **vector surface integral** of \( \mathbf{F} \) along \( \mathbf{X} \) is

\[
\int \int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} =
\]
6. Compute the scalar or vector surface integral where appropriate.

(a) \( \int \int_X z^3 \, dS \) where \( X : [0, 2\pi] \times [0, \pi] \rightarrow \mathbb{R}^3 \) is the parametrized sphere of radius \( a \)

(b) \( \int \int_X \mathbf{F} \cdot d\mathbf{S} \) where \( \mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z - 2y)\mathbf{k} \) where \( X(s, t) = (s \cos t, s \sin t, t) \) for \( 0 \leq s \leq 1 \) and \( 0 \leq t \leq 2\pi \)
(c) \[ \int_S xyz \, dS \] for \( S \) a closed cylinder with bottom given by \( z = 0 \) and top given by \( z = 4 \) and lateral surface given by \( x^2 + y^2 = 9 \) (orient \( S \) with outward normals)

(d) \[ \int_S y^3i \cdot d\mathbf{S} \] for \( S \) as in (c)

Stokes' and Gauss' Theorems (7.3)

Stokes' Theorem

Let \( S \) be a bounded, piecewise smooth, oriented surface in \( \mathbb{R}^3 \). Suppose that \( \partial S \) consists of finitely many piecewise \( C^1 \), simple, closed curves each of which is oriented consistently with \( S \). Let \( \mathbf{F} \) be a vector field of class \( C^1 \) whose domain includes \( S \). Then

\[
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} =
\]
7. Verify Stokes’ Theorem for the surface $S$ defined by $x^2 + y^2 + 5z = 1, z \geq 0$, oriented by upward normal, and $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + (x^2 + y^2)\mathbf{k}$

8. Use Stokes’ Theorem to find the work done by the vector field $\mathbf{F} = (xyz - e^x)\mathbf{i} - xyz\mathbf{j} + (x^2yz + \sin z)\mathbf{k}$ on a particle that moves along the line segments from $(0,0,0)$, then to $(1,1,1)$, then to $(0,0,2)$, then back to $(0,0,0)$. 
Gauss’ Theorem

Let $D$ be a bounded solid region in $\mathbb{R}^3$ whose boundary $\partial D$ consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point away from $D$. Let $\mathbf{F}$ be a vector field of class $C^1$ whose domain includes $D$. Then

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} =$$

9. Verify Gauss’ Theorem for the region $D = \{(x, y, z) \mid a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$ and the vector field $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$.

10. Use Gauss’ Theorem to find the volume of the solid region bounded by the paraboloids $z = 9 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 16$. 

11. Let \( n(x, y, z) \) be a unit normal to a surface \( S \). The directional derivative of a differentiable function \( f(x, y, z) \) in the direction of \( n \) is called the normal derivative of \( f \), denoted \( \partial f/\partial n \). From Theorem 6.2 of Chapter 2, we have

\[
\frac{\partial f}{\partial n} = \nabla \cdot n.
\]

(a) Let \( S \) denote the portion of the sphere \( x^2 + y^2 + z^2 = a^2 \) in the first octant \( (x, y, z \geq 0) \), oriented by the unit normal that points away from the origin. Let \( f(x, y, z) = \ln(x^2 + y^2 + z^2) \). Evaluate

\[
\oint_S \frac{\partial f}{\partial n} dS.
\]

(b) Let \( D \) denote the piece of the solid ball \( x^2 + y^2 + z^2 \leq a^2 \) in the first octant. Compute \( \iiint_D \nabla \cdot (\nabla f) dV \), where \( f \) is as in part (a).
(c) Apply Gauss’ Theorem to the integral in part (b), and reconcile your result with the answer in part (a).