(1) In Sec. 3, the reference [Mil90] Ch. III, Sec. 6, Rem. 6.1, which claims that the Galois finite étale cover $X_{\mathcal{H}(\ell^r)} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow X_{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q}$ in (3.2) has Galois group exactly $\mathcal{H}_{\ell}/\mathcal{U}(\ell^r)c$, is incorrect. (We thank Yihang Zhu for asking us about this reference and discussing with us about its validity.)

To see this, let $T := \ker(G \rightarrow G_c)$, which is a torus by the definition of $G_c$. (We will not even need to know that $T$ is $\mathbb{Q}$-anisotropic but $\mathbb{R}$-split.)

The claim in [Mil90] Ch. III, Sec. 6, Rem. 6.1 would be valid only if, for nontrivial $T$, the cardinalities of $T(\mathbb{Q}) \backslash T(\mathbb{A}^\infty)/\mathcal{H}_T$ (which is a finite set by [Bor63] Thm. 5.1) remain unchanged for all sufficiently small open compact subgroups $\mathcal{H}_T$ of $T(\mathbb{A}^\infty)$. This implies that the closure $\overline{T}(\mathbb{Q})$ of $T(\mathbb{Q})$ has finite index in $T(\mathbb{A}^\infty)$, but contradicts the fact that $\overline{T}(\mathbb{Q})$ has infinite index in $T(\mathbb{A}^\infty)$ for every nontrivial torus $T$ over $\mathbb{Q}$. (See [PR94] Prop. 7.13(2)], which explains that the same failure occurs, more generally, for algebraic groups over number fields that are connected but not simply-connected.)

This does not affect the construction of automorphic étale sheaves for representations of $G_c$, since all we need is that the Galois group is a quotient of $\mathcal{H}_\ell/\mathcal{U}_\ell(\ell^{r(m)})$ by construction, and admits $\mathcal{H}_\ell/\mathcal{U}_\ell(\ell^{r(m)})c$ as a quotient. In particular, in (3.3), the contraction product can be formed using the action of $\mathcal{H}_\ell/\mathcal{U}_\ell(\ell^{r(m)})$ instead, whose pullback to $X_{\mathcal{H}(\ell^{r(m)})} \otimes_{\mathbb{Z}} \mathbb{Q}$ is still isomorphic to $V_{0,\ell^m}$ by construction. The remainder of Sec. 3 is unaffected.

References


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